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A two-way paraxial system for simulation of wave backscattering by a random medium

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ABSTRACT

This paper presents a probabilistic analysis of an iterative two-way paraxial scheme for the simulation of wave propagation in anisotropic random media. This scheme has the computational cost of the standard one-way paraxial wave equation but has the accuracy of the full wave equation in a regime beyond the classical paraxial regime. More precisely, it accurately predicts the statistics of the reflected wave field. The accuracy depends on two parameters: the order of the iterative scheme and the ratio of the random backscattering intensity over the random forward-scattering intensity.

1. Introduction

Wave propagation in random media is numerically challenging when the propagation distance is large compared to the wavelength. The resolution of the full wave equation by difference methods is then in general computationally expensive since relatively fine scales have to be resolved. Fortunately, in many applications, such as ocean acoustics or atmospheric beam propagation, backscattering is negligible and the wave propagates mainly in a privileged direction, which allows for the reduction of the wave equation to the paraxial (or one-way) wave equation [19]. The time harmonic form of this equation takes the form of a Schrödinger equation with a random potential. This formulation is easy to solve numerically by finite-difference or split-step Fourier methods. It also allows for a theoretical analysis from the statistical point of view, since the solution is adapted to the filtration of the process that describes the fluctuations of the random medium. Indeed, the spatial argument corresponding to the privileged direction of propagation plays the role of "time" in the Schrödinger equation and also for the filtration of the medium fluctuations process.

In this paper we address situations in which backscattering is not negligible, which requires more elaborate schemes than the paraxial wave equation. After the pioneering work [7] an iterative two-way paraxial scheme was proposed to solve the full wave equation by an iteration of forward-going and backward-going paraxial wave equations in [15,16]. Comparisons

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between numerical simulations of the full wave and numerical simulations of the two-way paraxial system have shown very good agreement, with the overall conclusion being that the two-way paraxial scheme has the same numerical complexity as the standard one-way paraxial equation, but it also takes into account random backscattering. However, theoretical justifications are yet to be found. In particular, the role of the order of the iterative scheme is not clear in these first papers.

The purpose of this paper is to give theoretical arguments that show that the two-way paraxial system can indeed give accurate predictions for the transmitted and reflected waves in propagation regimes in which backscattering due to random medium fluctuations is significant. We analyze in detail the case of an anisotropic medium with the correlation length in the propagation direction much shorter than that in the transverse direction. The general motivation for numerical wave propagation in random media is to examine the effects that medium heterogeneity has on the propagating wave. In particular, the goal in imaging is to identify the macroscale features of the medium and it is important to analyze the role of the fine scale medium heterogeneities. These fine scale heterogeneities are usually very complicated and cannot be identified explicitly, but only through statistical means. What is relevant in this context is therefore the statistics of the transmitted and reflected wave fields. It is important to note that our goal is thus not to give arguments for a *strong* equivalence between the paraxial system and the wave equation, in the sense that the wave fields obtained from the two systems coincide (with some accuracy) for a given realization of the random medium. Rather, we seek to show that it is possible to prove a *weak* equivalence, in the sense that the statistics of the wave fields obtained from the two systems coincide. As expected, the degree of accuracy of the two-way iterative scheme depends on the order of the scheme (i.e. the number of iterations) and we clarify the connection between accuracy and order. We focus in particular our attention on the first-order and the second-order two-way paraxial equation for the reflected wave field and some associated applications.

The paper is organized as follows: in Section 2 we present the full wave equation in random medium and its reduction to the standard paraxial wave equation. In Section 3 we introduce the two-way scheme. The statistical analysis of the reflected wave field is carried out in Section 5. This analysis requires us to transform the boundary value problem satisfied by the reflected and transmitted wave fields into an initial-value problem by an invariant imbedding technique which is presented in Section 4. Finally we present numerical simulations in Section 6.

2. The wave decomposition

We consider linear acoustic waves propagating in a $(1+d)$-dimensional heterogeneous medium. The governing equations for the pressure field p and the $(1+d)$-dimensional velocity field $\boldsymbol{u}$ are

$$\rho(z,\boldsymbol{x})\frac{\partial \boldsymbol{u}}{\partial t}+\nabla p=\boldsymbol{0},\quad \frac{1}{K(z,\boldsymbol{x})}\frac{\partial p}{\partial t}+\nabla\cdot\boldsymbol{u}=0, \tag{1}$$

where ρ is the density of the medium, K is the bulk modulus of the medium, and $(z,\boldsymbol{x})\in\mathbb{R}\times\mathbb{R}^d$ are the space coordinates. We assume that a section of heterogeneous medium is sandwiched in between two homogeneous half-spaces and we consider the following model for the bulk modulus and density:

$$\frac{1}{K(z,\boldsymbol{x})}=\begin{cases}K_0^{-1} & \text{if } z\in(-\infty,0)\cup(L,\infty),\\ K_0^{-1}(1+\nu(z,\boldsymbol{x})) & \text{if } z\in(0,L),\end{cases}$$
$$\rho(z,\boldsymbol{x})=\rho_0,$$

where K_0 and ρ_0 are two positive constants, the homogenized parameters, and $\nu(z,\boldsymbol{x})$ is a centered random field that is bounded by a constant smaller than one. The acoustic wave equations can then be reduced to the inhomogeneous wave equation for the pressure field:

$$\Delta p-\frac{\rho_0}{K(z,\boldsymbol{x})}\frac{\partial^2 p}{\partial t^2}=0, \tag{2}$$

where Δ is the $(1+d)$-dimensional Laplacian. By taking a Fourier transform with respect to time

$$\check{p}(\omega,z,\boldsymbol{x})=\int p(t,z,\boldsymbol{x})e^{i\omega t}\,dt,\quad p(t,z,\boldsymbol{x})=\frac{1}{2\pi}\int\check{p}(\omega,z,\boldsymbol{x})e^{-i\omega t}\,d\omega,$$

we obtain the inhomogeneous Helmholtz equation

$$\Delta\check{p}+(1+\nu(z,\boldsymbol{x})\mathbf{1}_{(0,L)}(z))\frac{\omega^2}{c_0^2}\check{p}=0, \tag{3}$$

where we have introduced the background velocity $c_0=\sqrt{K_0/\rho_0}$. The Helmholtz equation is complemented by boundary values at the interfaces $z=0$ and $z=L$ given by the incoming field and also radiation conditions. These boundary and radiation conditions have a convenient representation if we decompose the solution of the wave equation into generalized right- and left-going mode amplitudes $\check{a}$ and $\check{b}$ [11,12]: