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High-accuracy large-step explicit Runge–Kutta (HALE-RK) schemes for computational aeroacoustics

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ABSTRACT

In many realistic calculations, the computational grid spacing required to resolve the mean flow gradients is much smaller than the grid spacing required to resolve the unsteady propagating waves of interest. Because of this, the high temporal resolution provided by existing optimized time marching schemes can be excessive due to the small time step required for stability in regions of clustered grid. In this work, explicit fourth-order accurate Runge–Kutta time marching schemes are optimized to increase the inviscid stability limit rather than the accuracy at large time steps. Single and multiple-step optimized schemes are developed and analyzed. The resulting schemes are validated on several realistic benchmark problems.

1. Introduction

The field of computational aeroacoustics (CAA) is focused on the accurate simulation of unsteady flow and noise [1,2]. To achieve this goal, highly accurate spatial differencing schemes (e.g., [1,3–5,7]), along with optimized time marching schemes (e.g., [4,8–10]) have been developed. To validate these schemes, a range of benchmark validation problems has been specified, and solutions made available [11–14]. As the CAA schemes have increased in capability, the validation problems have increased in complexity, incorporating realistic nonlinear flows about complex geometries.

As more complex problems are attempted, the underlying assumptions made in the development of the CAA schemes sometimes must be re-evaluated. For example, the optimized spatial differencing and time marching incorporated in CAA methods are designed to propagate accurately unsteady flow phenomena through non-uniform grids wrapped about complex geometries such as the stators in a turbojet engine. Experience has shown that the increased accuracy from the spatial differencing schemes is always beneficial, even near flow discontinuities. However, the use of high-accuracy explicit time marching schemes can result in an excessive number of time steps due to the stability limit on the time step size.

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Nomenclature

A	area of duct
CFL	Courant–Friedrichs–Lewy stability condition for explicit time marching schemes
k	physical wavenumber of propagating wave
$(k\Delta x)^*$	numerical wavenumber of spatial differencing scheme
λ	physical wavelength of propagating wave
ω	physical frequency of propagating wave
$\omega\Delta t$	numerical frequency of time marching scheme

In a time marching scheme for an unsteady flow problem, there are two time steps of interest. The first is the largest time step that can be taken while retaining an accurate unsteady solution (the accuracy limit). The second time step of interest is the largest time step that can be taken while retaining a stable calculation (the stability limit).

The inviscid stability limit on an explicit time marching scheme is directly related to the minimum time required for the fastest propagating wave to move from one grid point to the next (the CFL condition). In a time marching scheme using a single time step size throughout the computational domain, such as a Runge–Kutta time marching scheme, this results in the smallest grid spacing determining the time step for the entire calculation, regardless of whether the resulting temporal resolution is necessary for solution accuracy. One method to avoid this problem is to use a multi-step Adams–Bashforth time marching scheme, allowing different grid blocks to march at different time steps while retaining time accuracy [15]. Such methods are effective, but are hard to generalize for use on arbitrary flow problems. Implicit time marching schemes can also allow large time steps [16] at the price of sub iterations to converge the solution at each time step.

A popular explicit Runge–Kutta scheme is the classical four-stage fourth-order scheme, which has a stability limit of CFL = 2.83. In the past, several researchers have optimized Runge–Kutta schemes to increase their stability and/or accuracy. For steady-state calculations, Jameson [17] developed a five-stage second-order scheme, which has a stability limit of CFL = 4.0 but low accuracy for time marching. Hu et al. [8] introduced the low-dispersion and dissipation Runge–Kutta (LDDRK) schemes, which were optimized for accuracy to the stability limit. The most popular of these schemes is the two-step fourth-order RK56 scheme, which has a stability limit of CFL = 2.85. Recently, Calvo et al. [10] have introduced a six-stage fourth-order Runge–Kutta scheme that is partially optimized for accuracy and partially optimized for a large stability limit. The resulting scheme has a stability limit of CFL = 3.82.

In this work, six/seven stage fourth-order Runge–Kutta schemes are optimized to maximize the inviscid stability limit while retaining fourth-order accuracy. The effect of optimizing a single-step scheme is compared to the results obtained by optimizing two-step methods. The resulting schemes have inviscid stability limits of up to a CFL of 5.7, and have better accuracy than a classical fourth-order Runge–Kutta scheme.

2. Linear analysis of numerical schemes

In order to analyze the performance of a time-marching numerical method, a linear model equation for inviscid flow is used:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0. \tag{1}$$

A simple-harmonic solution to the equation is assumed:

$$u(x,t) = e^{i(kx-\omega t)}, \tag{2}$$

where,

$$\omega = ck. \tag{3}$$

This solution represents a simple-harmonic wave propagating at a speed of c in the positive x direction. In applying a numerical scheme to solve Eq. (1), the spatial and temporal derivatives are replaced by numerical approximations, each of which impacts the accuracy of the solution. In the case of a propagating wave, the errors can be classed as *dispersion* (a change in the propagation speed of the wave) and *dissipation* (a change in the amplitude of the wave).

The spatial derivative is considered first. The analytic result for the spatial derivative is:

$$\left.\frac{\partial u}{\partial x}\right|_{j,\text{analytic}} = iku_j = \frac{i(k\Delta x)}{\Delta x}u_j. \tag{4}$$

A finite-difference spatial derivative at grid point j on a uniform grid can be written as:

$$\left.\frac{\partial u}{\partial x}\right|_{j,\text{numerical}} = \frac{1}{\Delta x}\sum_{n=-M}^{N} a_n u_{j+n}. \tag{5}$$