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# An improved weighted essentially non-oscillatory scheme with a new smoothness indicator

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ABSTRACT

In this paper, we present a new smoothness indicator that evaluates the local smoothness of a function inside of a stencil. The corresponding weighted essentially non-oscillatory (WENO) finite difference scheme can provide the fifth convergence order in smooth regions, especially at critical points where the first derivative vanishes (but the second derivatives are non-zero). We provide a detailed analysis to verify the fifth-order accuracy. Some numerical experiments are presented to demonstrate the performance of the proposed scheme. We see that the proposed WENO scheme provides at least the same or improved behavior over the fifth-order WENO-JS scheme [10] and other fifth-order WENO schemes called as WENO-M [9] and WENO-Z [2], but its advantage seems more salient in two dimensional problems.



## 1. Introduction

The goal of this paper is to introduce an improved version of the fifth-order WENO finite difference scheme for the approximation of hyperbolic conservation laws in the form

$$\begin{aligned} &q_t + f(q)_x = 0, \quad x \in \mathbb{R},\ t \geqslant 0, \\ &q(x,0) = q_0(x), \end{aligned} \tag{1.1}$$

with proper boundary conditions. Here, the function $q = (q_1, \ldots, q_m)$ is an $m$-dimensional vector of conserved quantities and the flux $f(q)$ is a vector-valued function of $m$ components. Eq. (1.1) is called hyperbolic if all eigenvalues $\{\lambda^k(q)\}$ of the Jacobian matrix

$$A = \frac{\partial f}{\partial q},$$

of the flux function are real and the set of right eigenvectors is complete.

Since the (exact) solution of the hyperbolic Eq. (1.1) may develop discontinuities—shock and contact—in finite time, one needs shock-capturing schemes without creating spurious oscillations. Among many numerical schemes for solving (1.1), ENO (essentially non-oscillatory) [7,8,16,17] and Weight ENO (WENO) schemes [10,12] have been quite successful in applications with shocks, contact discontinuities, and complicated smooth solutions. The ENO scheme, which is a modification of

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the total variation diminishing (TVD) scheme [5], is designed to use an adaptive stencil based on the local smoothness such that it yields high-order accuracy where the function is smooth but avoids the Gibbs phenomena at discontinuities. To do this, a *smoothness indicator* of a solution is first determined over each stencil and then, by using this, the smoothest one is chosen from a set of candidate stencils. As a result, an ENO scheme obtains information from smooth regions and avoids spurious oscillations near discontinuities. The cell-average version of ENO schemes has the disadvantage of complicate transfers between point values and cell averages for a higher order accuracy in multi-spatial dimension problems. To eliminate these complications, efficient ENO schemes based on point values rather than cell averages were introduced by Shu and Osher [16,17]. Also, Serna and Marquina [14] introduced power ENO methods using an extended class of limiters to classical ENO schemes to improve the algorithmic behavior near discontinuities.

The WENO scheme, an improved version of the ENO technique using a cell-averaged approach, was introduced by Liu, Osher, and Chan [12]. It uses a nonlinear convex combination of all the candidate stencils to improve the accuracy of numerical fluxes without destroying non-oscillatory behaviors near discontinuities. This process is performed by weighting the contribution of the local flux according to its smoothness on each stencil such that the weight of the solution on a stencil containing a discontinuity is essentially zero. In doing this, the corresponding WENO scheme can achieve a high order accuracy without oscillations near discontinuities or sharp gradient regions. Specifically, the roles of smoothness indicators in WENO schemes are (1) improving the convergence order at the smooth regions, (2) decreasing the dissipation near discontinuities, and (3) maintaining the stability without destroying the essentially non-oscillatory behavior. In the paper [10], Jiang and Shu introduced a smoothness indicator, which is the sum of the normalized squares of the scaled $L^2-$norms of the all derivatives of the lower order polynomials, and extended a finite difference (flux) version of WENO schemes [12] to third- and fifth-order accurate methods (hereafter, denoted by WENO-JS). However, Henrick, Aslam, and Powers pointed out that the actual convergence rate of the fifth-order WENO-JS in [10] is less than fifth order for many problems [9]. They demonstrated that the smoothness indicator of WENO-JS was failed to recover the maximum order of the scheme at critical points where the third order derivatives do not simultaneously vanish with the first. In the same article, they introduced a new fifth-order WENO method (called WENO-M) fixing this problem by modifying the smoothness indicator of WENO-JS to satisfy sufficient criteria for fifth-order convergence. They devised a correcting mapping and applied it to the smoothness indicator of WENO-JS. Compared to the WENO-JS scheme, the WENO-M scheme can achieve the optimal convergence order at critical points of smooth parts, reduce the numerical dissipation, and obtain sharper results near discontinuities. In [2], Borges, Carmona, Costa, and Don introduced another version of the fifth-order WENO scheme (called WENO-Z) with a new smoothness indicator which is constructed by incorporating a global higher order smoothness measurement into the weights of the WENO-JS scheme. This method allows us to obtain the optimal convergence order at the critical points of smooth regions of the solution and captures shock in a physically sharp manner with maintaining stability and the essentially non-oscillatory behavior. The WENO-Z scheme has the same accuracy as the WENO-M scheme but generates improved results. Higher order WENO schemes also have been developed. Balsara and Shu [1] introduced WENO methods up to the eleventh order. In [4], Gerolymos, Sénéchal, and Vallet extended both the WENO-JS and WENO-M schemes up to the seventeenth order. In addition, Castro, Costa, and Don provided a closed-form formula for higher than fifth order smoothness indicators of the WENO-Z scheme in [3].

In this article, we derive a new method that measures the (local) smoothness of the numerical solution inside a stencil. The associated WENO scheme (hereafter, denoted by WENO-NS) can get the fifth convergence order in smooth regions, even at critical points where the first derivative vanishes (but the secod derivatives are non-zero). For the construction of the new smoothness indicator, we first introduce a generalized version of undivided difference that approximates the derivatives of the flux functions more accurately. Based on this approximation, a new smoothness indicator is constructed. An interesting feature is that there is a parameter that governs the tradeoff between the accuracies around the smooth region and discontinuous region. A good choice for the parameters can be interpreted as a balanced compromise between smoothness and discontinuity of the data. Some numerical experiments are presented to demonstrate the performance of the proposed scheme. The WENO-NS scheme provides at least similar or improved behavior over other fifth-order WENO schemes: WENO-JS, WENO-M and WENO-Z. Also, we will see that the advantage of the proposed scheme seems more salient in two dimensional problems.

The rest of this paper is organized as follows. Section 2 provides a brief review of the WENO schemes for one-dimensional scalar conservation laws. In Section 3, we provide a new smoothness indicator and the associated fifth-order WENO scheme. Section 4 presents some numerical results to demonstrate the ability of the proposed WENO scheme, and a conclusion is presented in Section 5.

## 2. Review of the WENO schemes

In this section, we briefly describe the fifth-order WENO finite difference scheme for one-dimensional scalar conservation laws. Let $\{I_j\}$ be a partition of a given domain with the $j$th cell $I_j := [x_{j-1/2}, x_{j+1/2}]$. The center of $I_j$ is denoted by $x_j = \frac{1}{2}(x_{j-1/2} + x_{j+1/2})$ and the value of a function $f$ at the location $x_j$ is denoted by using the subscript $j$, i.e., $f_j = f(x_j)$. In what follows, for practical use, we assume that the set $\{x_{j+1/2}\}_j$ is uniformly gridded. Then, the notation $\Delta x = x_{j+1/2} - x_{j-1/2}$ indicates the size of $I_j$.

The one-dimensional hyperbolic conservation laws in (1.1) can be approximated by a system of ordinary differential equations, where the spatial derivative has been replaced by a finite difference, so that it yields the semi-discrete form: