Journal of Computational Physics 232 (2013) 456–467

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Compact difference scheme for the fractional sub-diffusion equation with Neumann boundary conditions ☆

Jincheng Ren [a,b], Zhi-zhong Sun [a,*], Xuan Zhao [a]

[a] Department of Mathematics, Southeast University, Nanjing 210096, PR China
[b] Department of Mathematics, Shangqiu Normal University, Henan Shangqiu 476000, PR China

ARTICLE INFO

Article history:
Received 24 April 2012
Received in revised form 11 August 2012
Accepted 19 August 2012
Available online 31 August 2012

Keywords:
Fractional sub-diffusion equation
Compact difference scheme
Discrete energy method
Stability
Convergence
Neumann boundary conditions

ABSTRACT

An effective finite difference scheme is considered for solving the time fractional sub-diffusion equation with Neumann boundary conditions. A difference scheme combining the compact difference approach the spatial discretization and L_1 approximation for the Caputo fractional derivative is proposed and analyzed. Although the spatial approximation order at the Neumann boundary is one order lower than that for interior mesh points, the unconditional stability and the global convergence order $O(\tau^{2-\alpha}+h^4)$ in discrete L_2 norm of the compact difference scheme are proved rigorously, where τ is the temporal grid size and h is the spatial grid size. Numerical experiments are included to support the theoretical results, and comparison with the related works are presented to show the effectiveness of our method.

1. Introduction

Many phenomena in engineering and applied sciences can be described successfully by developing models using fractional calculus, such as materials and mechanics, signal processing, anomalous diffusion, biological systems, finance, hydrology and so on (cf. [1–7]). Fractional sub-diffusion equation is a class of anomalous diffusive systems, which is obtained by replacing the time derivative in ordinary diffusion by a fractional derivative of order α with $0<\alpha<1$. It is a known fact that the anomalous diffusion is characterized by a diffusion constant and the mean square displacement of diffusing species in the form

$$\langle x^2(t)\rangle \sim t^{\alpha}, \quad t\to\infty,$$

where $\alpha (0<\alpha<1)$ is the anomalous diffusion exponent.

Numerical approaches to different types of fractional diffusion models have been increasingly appearing in the literature. Recent work on numerical solutions for the fractional anomalous diffusion equation discussed here, can be found in [8–24]. Yuste and Acedo [8] and Yuste [9] presented an explicit scheme and weighted average finite difference methods for the time fractional diffusion equation and analyzed these two schemes' stability by the von Neumann method. Chen and Liu [10] constructed the difference scheme based on Grünwald-Letnikov formula and showed the stability and convergence of the difference scheme using the Fourier method for the fractional sub-diffusion equation. Zhuang et al. [11] introduced a new way

☆ The research is supported by National Natural Science Foundation of China (No. 11271068), the Research and Innovation Project for College Graduates of Jiangsu Province (No. CXZZ11_0134, CXLX11_0093) and Foundation for Key Teacher of Shangqiu Normal University (No. 2012GGJS14).
* Corresponding author. Tel.: +86 25 83793725; fax: +86 25 83792316.
E-mail addresses: renjincheng2001@126.com (J. Ren), zzsun@seu.edu.cn (Z.-z. Sun), xzhao@seu.edu.cn (X. Zhao).

0021-9991/$ - see front matter Crown Copyright © 2012 Published by Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jcp.2012.08.026

for solving sub-diffusion equation by integration of the original equation on the both sides to obtain an implicit numerical method. The stability and convergence of the scheme were proved by the energy method. Later, the same ideas were applied to solve the non-linear fractional reaction-subdiffusion process [12], anomalous subdiffusion equation with a nonlinear source term [13]. Sun and Wu [14] derived a fully discrete difference scheme for the fractional diffusion-wave equation and sub-diffusion equation, and proved that the schemes were uniquely solvable, unconditionally stable and convergent in maximum norm with the convergence order of $O(\tau^{3-\alpha}+h^2)$ and $O(\tau^{2-\alpha}+h^2)$, respectively. Yang et al. [16] proposed two numerical schemes to solve time–space fractional diffusion equation in two dimensions using either the finite difference method or the Laplace transform to handle the Caputo time fractional derivative. Lin and Xu [17] proposed an effective numerical method to solve the time fractional diffusion equation based on a finite difference scheme in time and Legendre spectral methods in space. Li and Xu [18] extended their previous work and proposed a spectral method in both temporal and spatial discretizations. Ji and Tang [19] and Zhang et al. [20] constructed local discontinuous Galerkin method and finite difference/element method for solving the fractional diffusion equations, respectively.

The complexity of the fractional differential equations comes from involving fractional derivatives that are nonlocal and have the character of history dependence and universal mutuality. It means that all previous solutions have to be saved to compute the solution at the current time level, which make the storage expensive. Due to the high spatial accuracy, the compact difference methods need few grid points to produce highly accuracy solution. Cui [25] considered a high-order finite difference scheme for solving the fractional anomalous sub-diffusion equation. The Grünwald formula was used to directly approximate the Riemann–Liouville fractional derivative in temporal direction and fourth order compact difference scheme for the spatial discretization, where the convergence order was $O(\tau+h^4)$ in discrete L_2 norm. Chen et al. [26] also focused on some high accuracy numerical methods. By the similar discretization approach, a scheme with convergence order $O(\tau+h^4)$ in L_2 norm was also obtained for the variable-order anomalous differential equation using the Fourier method. Gao and Sun [27] proposed a compact difference scheme for the time fractional sub-diffusion equation, and proved that the scheme was unconditionally stable and convergent in maximum norm with the convergence order of $O(\tau^{2-\alpha}+h^4)$. Du et al. [28] derived a compact difference scheme for the time fractional diffusion-wave equation based on L_1 approximation. In [29], Zhang et al. constructed a Crank–Nicolson-type difference scheme and a compact difference scheme for solving the time fractional sub-diffusion equation with Riemann–Liouville fractional derivative, respectively. They proved that the two difference schemes were unconditionally stable and the numerical solution was convergent in the maximum norm. Recently, Zhang et al. [30] and Cui [31] constructed alternating direction implicit scheme and compact alternating direction implicit scheme for solving the two-dimensional time fractional sub-diffusion equation, respectively.

The works mentioned above are dealing with the Dirichlet boundary conditions, where no boundary discretization errors are involved. However, for Neumann boundary value problem, the discretization of boundary conditions must be dealt with carefully to match the global accuracy. Langlands and Henry [32] developed an implicit difference scheme with convergence order $O(\tau+h^2)$ based on L_1 approximation for the Riemann–Liouville fractional derivative and numerically verified the unconditional stability of difference scheme but without global convergence analysis. Recently, Zhao and Sun [33] proposed a Box-type scheme for solving a class of fractional sub-diffusion equation with Neumann boundary conditions. Since many application problems in science and engineering involve Neumann boundary conditions [32,34,35], such as zero flow or specified flow flux condition. Thus, it is very desirable to use high-order algorithms for efficient computations of the numerical solution of this kind of problem. This motivates us to consider the compact difference method for spatial discretization.

The novelty of this paper is to construct effective and fast numerical methods for the time fractional sub-diffusion equation with Neumann boundary conditions and establish corresponding error estimates. In order to reduce the storage requirements, we adopt fourth order compact difference method for spatial approximation (cf. [25–27]), which needs fewer grid points to produce highly accurate solution. Using the L_1 approximation proposed by Xu [17] and Sun [14] to deal with temporal Caputo fractional derivative. The truncation error of the boundary approximation is with the order of $O(\tau^{2-\alpha}+h^3)$, which is one order lower in spatial direction than that for interior mesh points, where the approximating order is $O(\tau^{2-\alpha}+h^4)$. By using discrete Sobolev inequality (Lemma 3.1) and some novel techniques, the compact difference scheme is unconditionally stable and the global convergence order $O(\tau^{2-\alpha}+h^4)$ is proved rigorously.

The rest of the paper is organized as follows. In Section 2, we first transform the original problem to another equivalent form and give some auxiliary lemmas, then the derivation of the difference scheme is presented. In Section 3, using discrete Sobolev inequality and the energy method, the stability and convergence are analyzed. In Section 4, numerical experiments are carried out to support the theoretical analysis and to show the efficiency of the compact difference scheme and compare our method with the methods proposed in [33]. Some comments are presented in the concluding section.

2. A compact finite difference scheme for the fractional sub-diffusion equation

2.1. Notations and auxiliary lemmas

We give some notations and auxiliary lemmas, which will be used in the construction of the compact finite difference scheme.

We consider the following one-dimensional fractional sub-diffusion equation with the zero flux boundary conditions (cf. [32,33])