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All-at-once solution of time-dependent Stokes control

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ABSTRACT

The solution of time-dependent PDE-constrained optimization problems subject to unsteady flow equations presents a challenge to both algorithms and computing power. In this paper we present an all-at-once approach where we solve for all time-steps of the discretized unsteady Stokes problem at once. The most desirable feature of this approach is that for all steps of an iterative scheme we only need approximate solutions of the discretized Stokes operator. This leads to an efficient scheme which exhibits mesh-independent behaviour.

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1. Introduction

The solution of complex flow problems is one of the most interesting and demanding problems in applied mathematics and scientific computing. Over the last decades the numerical solution of problems such as Stokes flow has received a lot of attention both from applied scientists and mathematicians alike. The discretization of the Stokes equation via finite elements [13,1,10] as well as the efficient solution of the corresponding linear systems in saddle point form [13,48,43,3] are well established. In recent years, with the advances of computing power and algorithms, the solution of optimal control problems with partial differential equation (PDE) constraints such as Stokes or Navier–Stokes flow problems have become a topic of great interest [22,25,38,7,12].

In this paper, we want to address the issue of efficiently solving the linear systems that arise when the optimal control of the time-dependent Stokes problem is considered. We here want to employ the so-called all-at-once approach, which is a technique previously used in [23,24,5,44]. In detail, the discretization of the problem is constructed in the space–time domain and then solved for all time-steps at once. One of the advantages of this approach is that the PDE-constraint does not need to be satisfied until convergence of the overall system is reached. We will come back to this later.

One of the crucial ingredients to derive efficient preconditioners that show robustness with respect to the important problem parameters such as the mesh-parameter and the regularization parameter is the construction of efficient Schur-complement approximations. Recently, Pearson and Wathen [34] introduced an approximation that satisfies these criteria for the Poisson control problem. We here extend their result to a time-dependent problem subject to Stokes equation. In contrast to [34] the new approximation cannot simply be handled by a multigrid scheme but has to be embedded in a stationary iteration due to the indefiniteness of the discrete Stokes system.

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The paper is organized as follows, we first discuss the control problem and how it can be discretized. In Section 3 we discuss the choice of the Krylov solver that should be employed. We then discuss the preconditioners for the various parts of the saddle point problem. This is followed by numerical experiments for two different objective functions in both two and three space dimensions and time.

2. Problem and discretization

In the following we consider the tracking-type functional

$$J(y,u) = \frac{1}{2}\int_0^T \int_{\Omega_1} (y-\bar{y})^2 dxdt + \frac{\beta}{2}\int_0^T \int_{\Omega_2} (u)^2 dxdt + \frac{\gamma}{2}\int_{\Omega_1} (y(T)-\bar{y}(T))^2 dx \tag{1}$$

where $\Omega_{1/2} \subseteq \Omega$ are bounded domains in $\mathbb{R}^d$ with $d = 2,3$. Additionally, for the state y and the control u the time-dependent Stokes equation has to be satisfied

$$y_t - \nu\triangle y + \nabla p = u \quad \text{in} \quad [0,T]\times\Omega \tag{2}$$

$$-\nabla\cdot y = 0 \quad \text{in} \quad [0,T]\times\Omega \tag{3}$$

$$y(t,\cdot) = g(t) \quad \text{in} \quad \partial\Omega, t\in[0,T] \tag{4}$$

$$y(0,\cdot) = y^0 \quad \text{in} \quad \Omega, \tag{5}$$

with y the state representing the velocity and p the pressure. Here, $\bar{y}$ is the so-called desired (velocity) state. The goal of the optimization is to compute the control u in such a way that the velocity field y will be as close as possible to $\bar{y}$. One might impose additional constraints both on the control u and the state y. One of the most common constraints in practice are the so-called box constraints given by

$$u_a \leqslant u \leqslant u_b \quad \text{and} \quad y_a \leqslant y \leqslant y_b,$$

which will not be discussed further (see [45,32] for simpler PDEs).

There are two techniques used to solve the above problem. The first is the so-called *Discretize-then-Optimize* approach, where we discretize the objective function to get $J_h(y,u)$ and also discretize the PDE (Eqs. (2)–(5) written as $\mathcal{B}_h(y,u) = 0$. This allows us to form the discrete Lagrangian

$$\mathcal{L}_h(y,u) = J_h(y,u) + \lambda^T \mathcal{B}_h(y,u),$$

stationarity conditions for which would lead to a system of first order or KKT conditions. The second approach follows a *Optimize-then-Discretize* principle where we write Eqs. (2)–(5) in the form $\mathcal{B}(y,u) = 0$ and then formulate the Lagrangian of the continuous problem as

$$\mathbf{L}(y,u) = J(y,u) + \langle \mathcal{B}(y,u), \lambda\rangle$$

where $\langle\cdot,\cdot\rangle$ defines a duality product (see [26] for details). Based on the continuous Lagrangian, first order conditions are considered, which are then discretized and solved. There is no recipe as to which of these approaches has to be preferred (see the discussion in [25]). Recently, discretization schemes have been devised so that both approaches lead to the same discrete optimality system (e.g [23]).

We begin by considering the first order conditions of the above infinite dimensional problem. We obtain the forward problem described in (2)–(5) from $\mathbf{L}_\lambda = 0$, the relation

$$\beta u + \lambda = 0 \tag{6}$$

follows from $\mathbf{L}_u = 0$ and is usually referred to as the gradient equation, and we obtain also the adjoint PDE

$$-\lambda_t - \nu\triangle\lambda + \nabla\xi = y - \bar{y} \quad \text{in} \quad [0,T]\times\Omega \tag{7}$$
$$-\nabla\cdot\lambda = 0 \quad \text{in} \quad [0,T]\times\Omega \tag{8}$$
$$\lambda(t,\cdot) = 0 \quad \text{on} \quad \partial\Omega, t\in[0,T] \tag{9}$$
$$\lambda(0,\cdot) = \gamma(y(T)-\bar{y}(T)) \quad \text{in} \quad \Omega, \tag{10}$$

from $\mathbf{L}_y = 0$. For more information see [47,46,23,24].

The question is now whether we can find a discretization scheme such that the Discretize-then-Optimize and the Optimize-then-Discretize approach coincide. We start the Discretize-then-Optimize approach by using a backward Euler scheme in time to obtain for the forward Stokes problem

$$\frac{y_k - y_{k-1}}{\tau} - \nu\triangle y_k + \nabla p_k = u_k \tag{11}$$
$$-\nabla\cdot y_k = 0 \tag{12}$$

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