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Spectral (finite) volume method for conservation laws on unstructured grids VI: Extension to viscous flow

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Abstract

In this paper, the spectral volume (SV) method is extended to solve viscous flow governed by the Navier–Stokes equations. Several techniques to discretize the viscous fluxes have been tested, and a formulation similar to the local discontinuous Galerkin (DG) approach developed for the DG method has been selected in the present study. The SV method combines two key ideas, which are the bases of the finite volume and the finite element methods, i.e., the physics of wave propagation accounted for by the use of a Riemann solver and high-order accuracy achieved through high-order polynomial reconstructions within spectral volumes. The formulation of the SV method for a 2D advection-diffusion equation and the compressible Navier–Stokes equations is described. Accuracy studies are performed using problems with analytical solutions. The solver is used to compute laminar viscous flow problems to shown its potential.

MRI: 65M60; 65M70; 35L65

Keywords: High-order; Unstructured grid; Spectral finite volume; Navier–Stokes equations

1. Introduction

We continue the development of the spectral (finite) volume (SV) method for hyperbolic conservation laws on unstructured grids following the basic formulation [27], development for two-dimensional scalar conservation laws [28], one-dimensional systems and partition optimization [29], two-dimensional systems [30], and three-dimensional linear systems [14]. In the present study, the SV method is extended to compute viscous flows governed by the Navier–Stokes equations. The SV method belongs to a general class of Godunov-type finite volume method [11,26], which has been under development for several decades, and has become the state-of-the-art for the numerical solution of hyperbolic conservation laws. For a more detailed review of

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the literature on the Godunov-type method, refer to [27], and the references therein. Some widely used numerical methods for conservation laws such as the k-exact finite volume [2,10], the essentially non-oscillatory (ENO) [12,1], and weighted ENO [13] methods are also Godunov-type methods. The SV method is also related to a popular finite-element method for hyperbolic conservation laws, the discontinuous Galerkin (DG) method [5–8] in that multiple degrees of freedom are used in a single element. Comparisons between the DG and SV methods have been made recently [24,33]. The SV method avoids the volume integral required in the DG method. However, it does introduce more interfaces where more Riemann problems are solved. For 2D Euler equations, both methods seem to achieve similar efficiency [24]. Both the DG and SV methods are capable of achieving the optimal order of accuracy. The DG method usually has a lower error magnitude, but the SV method allows larger time steps. Due to its inherent property of subcell resolution, the SV method appears to capture discontinuities with a higher resolution than the DG method.

Ultimately, we wish to apply the SV method to perform large eddy simulation and direct numerical simulation of turbulent flow for problems with complex geometries. To achieve this goal, we must first find a technique to properly discretize the second-order viscous terms. In the second-order finite volume method, the solution gradients at an interface are usually computed by averaging the gradients of the neighboring cells sharing the face. For higher-order elements, special care has to be taken in computing the solution gradients. For example, Cockburn and Shu developed the so-called local discontinuous Galerkin method (LDG) to treat the second-order viscous terms and proved stability and convergence with error estimates [9] motivated by the successful numerical experiments of Bassi and Rebay [3]. Baumann and Oden [4], Oden et al. [16] introduced various penalty-type methods for the discretization of second-order viscous terms. Riviere et al. [17] analyzed three discontinuous Galerkin approximations for solving elliptic problems in two or three dimensions. More recently, Shu [22] summarized three different formulations for the diffusion equation, and Zhang and Shu [34] performed a Fourier type analysis for these three formulations. Recently, several formulations based on the successful LDG and penalty-type approaches have been developed and analyzed for the SV method using the 1D pure diffusion equation [25]. Three SV formulations, i.e., naïve formulation, local SV (LSV) and penalty SV (PSV) approaches, are tested. In the naïve formulation, the gradients on a face are obtained by averaging the gradients from the two cells sharing the face. It was found that the naïve formulation converges to the wrong solution, while the LSV and the PSV approaches are consistent, stable and convergent. It was shown that the LSV method achieved the optimal order of accuracy, i.e., $(k+1)$th order for degree k polynomial reconstructions. The PSV approach, however, achieved only kth order accuracy if k is even. Therefore, the LSV approach is selected for the extension to the Navier–Stokes equations in the present study. Before we attempt to solve the full 2D Navier–Stokes equations, the LSV formulation is further tested on 1D (both linear and non-linear) and 2D convection-diffusion equations.

The paper is therefore organized as follows. In Section 2, we describe the spectral volume formulation for the 2D convection-diffusion equation. The degeneration from 2D to 1D should be obvious. After that, the extension of the SV method to the Navier–Stokes equations is presented in Section 3. Section 4 presents numerical results including accuracy studies for the convection-diffusion equation. In addition, computations of laminar flows over a flat plate, a circular cylinder, and a NACA0012 airfoil are carried out, and results are compared with benchmark computations. Finally, conclusions and some possible future work are summarized in Section 5.

2. Spectral volume formulation for 2D convection-diffusion equation

For the sake of simplicity, the following 2D convection-diffusion equation is considered first in domain Ω with proper initial and boundary conditions

$$\frac{\partial u}{\partial t} + \nabla \cdot (\boldsymbol{\beta} u) - \nabla \cdot (\mu \nabla u) = 0, \tag{2.1}$$

where $\boldsymbol{\beta}$ is the convective velocity vector and μ is a positive diffusion coefficient. The computational domain Ω is discretized into N non-overlapping triangular cells. These cells, denoted as S_i, are called spectral volumes (SVs) in the SV method, i.e., $\Omega = \bigcup_{i=1}^{N} S_i$. An SV is further partitioned into a set of structured subcells, called control volumes (CVs), depending on the degree of the reconstruction polynomial. The partitions to be used in