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Explicit and implicit FEM-FCT algorithms with flux linearization

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ABSTRACT

A new approach to the design of flux-corrected transport (FCT) algorithms for continuous (linear/multilinear) finite element approximations of convection-dominated transport problems is pursued. The algebraic flux correction paradigm is revisited, and a family of nonlinear high-resolution schemes based on Zalesak's fully multidimensional flux limiter is considered. In order to reduce the cost of flux correction, the raw antidiffusive fluxes are linearized about an auxiliary solution computed by a high- or low-order scheme. By virtue of this linearization, the costly computation of solution-dependent correction factors is to be performed just once per time step, and there is no need for iterative defect correction if the governing equation is linear. A predictor–corrector algorithm is proposed as an alternative to the hybridization of high- and low-order fluxes. Three FEM-FCT schemes based on the Runge–Kutta, Crank–Nicolson, and backward Euler time-stepping are introduced. A detailed comparative study is performed for linear convection–diffusion equations.

1. Introduction

Numerical solutions to convection-dominated flow problems are frequently corrupted by spurious oscillations or excessive numerical diffusion. The first scheme to ensure positivity/monotonicity even in the limit of pure convection was the *flux-corrected transport* (FCT) algorithm of Boris and Book [2,3].

The basic idea behind the classical FCT methodology is as follows:

(1) Advance the solution in time by a low-order scheme that incorporates enough *numerical diffusion* to suppress undershoots and overshoots.
(2) Correct the solution using *antidiffusive fluxes* limited in such a way that no new maxima or minima can form and existing extrema cannot grow.

Predictor–corrector algorithms of this kind can be classified as diffusion–antidiffusion (DAD) methods [7]. The job of the numerical diffusion built into the low-order scheme is to enforce the positivity constraint and provide good phase accuracy. The limited antidiffusive correction is intended to reduce the amplitude errors in a local extremum diminishing manner.

A more general approach to the design of high-resolution schemes is based on blending (hybridization) of high- and low-order flux approximations. As a rule of thumb, the former is supposed to be used in regions of smoothness and the latter in

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the neighborhood of steep fronts. The weighting factor for the hybrid flux approximation is chosen so as to enforce physical or mathematical constraints related to some known properties of analytical solutions (positivity, monotonicity, nonincreasing total variation). A fully multidimensional FCT algorithm of this type was proposed by Zalesak [33] who constrained the positive and negative antidiffusive fluxes so as to control the net increment to the solution value at each grid point. We refer to [34] for a detailed presentation of the underlying design philosophy and further developments.

Following the advent of FCT in the 1970s, many other high-resolution schemes have been developed and backed by mathematical theory. Harten's *total variation diminishing* (TVD) methods [11] are also based on an algebraic hybridization of high- and low-order fluxes. Schemes like MUSCL, PPM, and ENO/WENO represent higher-order extensions of the Godunov method [9] that involve polynomial reconstruction from cell averages and slope limiting [1]. This geometric approach has become very popular in the context of finite difference, finite volume, and discontinuous Galerkin (DG) methods. However, slope-limited polynomial reconstruction has no natural counterpart in the realm of continuous (linear and multilinear) finite elements, whereas 'algebraic' flux correction of FCT or TVD type is still feasible [20].

The development of high-resolution FEM on the basis of FCT dates back to the explicit algorithms of Parrott and Christie [27] and Löhner *et al.* [23,24]. Several implicit FEM-FCT schemes were published by the author and his coworkers [16,18,19,26]. The rationale for the use of an implicit time discretization stems from the fact that the CFL stability condition becomes prohibitively restrictive in the case of strongly nonuniform velocity fields and/or locally refined meshes. Woodward and Colella [31, p. 119] conclude that "adaptive grid schemes have a major drawback – they demand an implicit treatment of the flow equations." This statement reflects a widespread prejudice that implicit schemes are computationally expensive. As a matter of fact, the cost of an implicit algorithm depends on the choice of iterative methods, parameter settings, and stopping criteria. If the time step is very small, then a good initial guess is available and the sparse linear system can be solved with 1–2 iterations of the Jacobi or Gauß-Seidel method. Thus, the cost *per time step* approaches that of an explicit finite difference or finite volume scheme. As the time step increases, so does the number of iterations, and more sophisticated linear algebra tools (smoothers, preconditioners) may need to be employed.

If the antidiffusive fluxes depend on the unknown solution, an iterative solution strategy is adopted. In essence, the nonlinear algebraic system is replaced by a sequence of linearized ones in which the antidiffusive term is evaluated using the previously computed data. Sometimes, too many flux/defect correction cycles are required to obtain a converged solution, especially if the Courant number is large and the contribution of the consistent mass matrix cannot be neglected. The use of a discrete Newton method [26] makes it possible to accelerate convergence but the computational cost per time step is still rather high as compared to that of a fully explicit algorithm. This is unacceptable since the time step for FCT must be relatively small for accuracy reasons.

In order to reduce the cost of implicit flux correction, it is possible to compute the (unconstrained) high-order solution and use it to linearize the raw antidiffusive flux [18,23,26]. However, an implicit computation of the high-order predictor is expensive (or even impossible) due to the unfavorable properties of the discrete transport operator. In the present paper, we linearize the antidiffusive flux about the end-of-step solution computed by an explicit or implicit low-order scheme. This approach proves more efficient and simplifies the design of characteristic FCT schemes for nonlinear hyperbolic systems such as the Euler equations of gas dynamics. Furthermore, we apply limited antidiffusion to the low-order solution instead of modifying the algebraic system and solving it again. That is, we go back to the roots of FCT and adopt a predictor–corrector strategy of diffusion–antidiffusion type. Reportedly, such algorithms possess better phase accuracy than those based on hybridization [7].

Benchmark problems from [22] are used to evaluate the performance of explicit and implicit FEM-FCT schemes with flux linearization about the low-order predictor. In the explicit case, the use of a TVD Runge–Kutta time-stepping method [10] ensures positivity preservation, whereas no such proofs are available for the classical FEM-FCT algorithm [23,24] based on a two-step Taylor–Galerkin method. The numerical study to be presented demonstrates that the linearized Crank–Nicolson and backward Euler FCT schemes are 2–30 times faster than their nonlinear counterparts. Moreover, the computational cost per time step is comparable to that for Runge–Kutta FCT, although no attempt was made to optimize the parameter settings for the iterative solver.

2. Design criteria

As a model problem, consider a first-order hyperbolic equation that describes conservation of a scalar quantity u in a bounded domain Ω

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{f}(u) = 0 \quad \text{in } \Omega, \tag{1}$$

where $\mathbf{f}(u)$ is an inviscid flux function. A boundary condition is prescribed at the inlet Γ_-, where the velocity vector $\mathbf{v} = \mathbf{f}'(u)$ is pointing into Ω

$$u(\mathbf{x}) = g(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_-. \tag{2}$$

The initial condition for the problem to be solved is given as a function of $\mathbf{x}$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega. \tag{3}$$