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Local–global multiscale model reduction for flows in high-contrast heterogeneous media

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ABSTRACT

In this paper, we study model reduction for multiscale problems in heterogeneous high-contrast media. Our objective is to combine local model reduction techniques that are based on recently introduced spectral multiscale finite element methods (see [19]) with global model reduction methods such as balanced truncation approaches implemented on a coarse grid. Local multiscale methods considered in this paper use special eigenvalue problems in a local domain to systematically identify important features of the solution. In particular, our local approaches are capable of homogenizing localized features and representing them with one basis function per coarse node that are used in constructing a weight function for the local eigenvalue problem. Global model reduction based on balanced truncation methods is used to identify important global coarse-scale modes. This provides a substantial CPU savings as Lyapunov equations are solved for the coarse system. Typical local multiscale methods are designed to find an approximation of the solution for any given coarse-level inputs. In many practical applications, a goal is to find a reduced basis when the input space belongs to a smaller dimensional subspace of coarse-level inputs. The proposed approaches provide efficient model reduction tools in this direction. Our numerical results show that, only with a careful choice of the number of degrees of freedom for local multiscale spaces and global modes, one can achieve a balanced and optimal result.

1. Introduction

The high degree of variability and multiscale nature of heterogeneous porous media properties cause significant challenges for forward simulations. For this reason, some model reduction techniques are needed. Many multiscale and homogenization type methods are developed to provide an accurate representation of the solution. These approaches (e.g., [30,1,2,4,5,11,12,17,31,13,40,14,18,24,25,34,36,39]) approximate the effects of the fine-scale features and attempt to capture their effects on a coarse grid via localized basis functions. The main idea of the local multiscale methods is to construct basis functions that are used to approximate the solution on a coarse grid. Multiscale methods can be considered as local model reduction techniques that construct an approximation of the solution on a coarse grid for arbitrary coarse-level inputs. Our main goal is to use local model reduction techniques together with global ones in achieving efficient reduced-order approximations for input–output maps.

One of the earlier approaches in this direction is taken within the system theory framework [37,29,10,32,3]. In this setting, the system is written in terms of a mapping from input to output where the latter represents the quantity of interest for

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assessing the accuracy of the approximation. Input signals are regarded as quantities that we are controlling and output signals are the quantity that we are measuring. Characterizing the system by its input–output relation leads to the idea of state-space representation of the dynamical system, and in turn, to approximations by projecting the states to a smaller state-space dimension. The central issue in model reduction is, therefore, how to obtain efficiently the projection matrices, so that we preserve in the reduced-order models a small set of states that are relevant to the input–output behavior of the system. Approaches such as balanced truncation and proper orthogonal decompositions (POD) type techniques are proposed to perform global model reduction.

In this paper, we discuss how to combine efficient multiscale methods on a coarse grid with global model reduction techniques. We propose a model reduction technique that combines both local and global model reduction tools. We use the recently introduced local model reduction tools [19] that allow systematically identifying important degrees of freedom of the solution. These approaches allow lumping many degrees of freedom that can be localized as in homogenization. These local reduced models are combined with model reduction techniques on a coarse grid (that is inexpensive) to achieve an ultimate reduced model with fewer degrees of freedom. The resulting reduced model provides robust approximation of the global system at a substantially lower cost when the input belongs to a subset of all coarse-level functions.

In this paper, the balanced truncation method is used to reduce the global coarse-scale model. In the balanced truncation framework, one is interested in truncating simultaneously the states that are weakly controllable and weakly observable. This is accomplished by solving (coarse-scale) Lyapunov equations for the observability and controllability Gramians and computing the eigenvalues and eigenvectors of the product of Gramians. In the balanced truncation, one can obtain a priori bound for the error from the full system to the reduced system. Combining this bound with a priori error bound for coarse-grid multiscale approximation, we can achieve a reduced model with a prescribed accuracy. The accuracy of the proposed approach can be improved by adding local degrees of freedom as well as global degrees of freedom simultaneously. This is discussed in the paper.

We note that the use of approximate models in global model reduction methods such as within balanced truncation or POD approaches are not new. In many previous findings, various proxy models are used to make the model reduction affordable. Our main contribution is the use of systematic local model reduction methods that allow adding new degrees of freedom on a coarse-grid level. Moreover, our local approaches are capable of approximating the solution for any coarse-level inputs and thus the proposed methods will be efficient when the input belongs to a smaller dimensional space, e.g., the forcing or boundary conditions are defined only on a few coarse-grid blocks. Furthermore, our local approaches provide minimum degrees of freedom for high-contrast multiscale flow problems (see [19]), and thus the overall approach is efficient. The error estimates both on local and global model reduction methods allow obtaining a priori error estimates for the overall method.

Our approaches share similarities with reduced basis approaches [8,26,27,35] that can be considered as an umbrella approach for the techniques proposed here. The main idea of reduced basis approach is to select a number of snapshots that can be used to construct a basis for the solution space. In our approach, these snapshots are chosen on a coarse grid with some special choices of multiscale interpolation operators. In a paper by Krogstad [33], local POD techniques in combination with global solutions are used to construct multiscale basis functions. These methods approximate the local features of the solution using both global and local problems. These approaches differ from the ones proposed in our paper. In [16], the authors apply POD type methods together with coarse-scale models based on single-phase upscaling; however, these techniques only use single-phase permeability upscaling and the issues of enriching coarse spaces to balance the global and local model reduction errors are not investigated.

We present numerical results for flow problems in high-contrast multiscale media. We vary the dimensions of both local coarse spaces and global reduced space. We present the errors between fine-scale solution and corresponding coarse-scale solution and between coarse-scale solution and corresponding reduced coarse-scale solution obtained using some number of important modes of balanced truncation method. Our numerical results show that these errors can be comparable suggesting one needs to carefully select the coarse space when performing model reduction on the coarse system.

The paper is organized as follows. In the next section, we discuss the problem setting and local multiscale approximations. We present some preliminaries in Section 2. Section 3 is devoted to local–global multiscale model reduction. In particular, some remarks are discussed in Section 3.1. In Section 4, we present numerical results. Conclusions and final comments are presented in Section 5.

2. Preliminaries

We will demonstrate our methodology on an example of time-dependent single-phase flow, though the proposed method can be extended to other equations. For simplicity, we consider the following system where, for given input $u \in \mathbb{R}^m$, we seek of state $p(t,\cdot) \in H^1(D)$

$$\frac{\partial p}{\partial t} = \mathrm{div}(\kappa(x)\nabla p) + f(u) \in D. \tag{1}$$

Here $D \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) and for every input (or control) $u \in \mathbb{R}^m$, the forcing $f(u)$ is a linear functional of u and it is assumed to be square integrable. In many applications, we are interested in achieving model reduction that provide an accurate