Journal of Computational Physics 228 (2009) 1903–1915

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Method of fundamental solutions with optimal regularization techniques for the Cauchy problem of the Laplace equation with singular points

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ARTICLE INFO

Article history:
Received 8 March 2008
Received in revised form 13 November 2008
Accepted 16 November 2008
Available online 28 November 2008

Keywords:
Cauchy problem
Inverse problem
Laplace equation
L-curve
Method of fundamental solutions
Singular point
Tikhonov regularization

ABSTRACT

The purpose of this study is to propose a high-accuracy and fast numerical method for the Cauchy problem of the Laplace equation. Our problem is directly discretized by the method of fundamental solutions (MFS). The Tikhonov regularization method stabilizes a numerical solution of the problem for given Cauchy data with high noises. The accuracy of the numerical solution depends on a regularization parameter of the Tikhonov regularization technique and some parameters of the MFS. The L-curve determines a suitable regularization parameter for obtaining an accurate solution. Numerical experiments show that such a suitable regularization parameter coincides with the optimal one. Moreover, a better choice of the parameters of the MFS is numerically observed. It is noteworthy that a problem whose solution has singular points can successfully be solved. It is concluded that the numerical method proposed in this paper is effective for a problem with an irregular domain, singular points, and the Cauchy data with high noises.

1. Introduction

Many kinds of inverse problems have recently been studied in science and engineering. The Cauchy problem of an elliptic partial differential equation is a well known inverse problem. The Cauchy problem of the Laplace equation is an important problem which can be applied to the inverse problem of electrocardiography [2]. Onishi et al. [11] proposed an iterative method for solving the Cauchy problem of the Laplace equation. This method reduces the original inverse problem to an iterative process which alternatively solves two direct problems. This method, called the adjoint method in the papers [7,14], can solve various inverse problems by applying many kinds of numerical methods for solving partial differential equations, such as the finite difference method (FDM), the finite element method (FEM), and the boundary element method (BEM). The convergence of this method for the Cauchy problem of the Laplace equation has been obtained [13].

The method of fundamental solutions (MFS) is effective for easily and rapidly solving the elliptic well-posed direct problems in complicated domains. Mathon and Johnston [10] first showed numerical results obtained by the MFS. The papers [1,9] discuss some mathematical theories on the MFS. Both of the BEM and the MFS are well known boundary methods, which discretize original problems based on the fundamental solutions. The MFS does not require any treatments for the singularity of the fundamental solution, while the BEM requires singular integrals. The MFS is a true meshless method, and can easily be extended to higher dimensional cases.

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2008.11.018

Wei et al. [16] applied the MFS to the Cauchy problems of elliptic equations. This method uses the source points distributed outside the domain. The accuracy of numerical solutions depends on the location of the source points. They numerically showed the relation between the accuracy and the radius of a circle where the source points are distributed. But, the relation between the accuracy and the number of source points has not clearly been given, yet.

Many researchers have solved the Cauchy problem with various methods. However, to our knowledge, the conventional methods cannot solve a problem whose solution has singular points outside the computational domain (see [8,15] for example). Using the FDM or the spectral collocation method in multiple-precision arithmetic, we cannot successfully solve a problem such that the exact solution is unbounded outside the computational domain.

In this paper, we use the MFS to directly discretize the Cauchy problem of the Laplace equation. This is an ill-posed problem, where the solution has no continuous dependence on the boundary data. Namely, a small noise contained in the given Cauchy data has a possibility to affect sensitively on the accuracy of the solution. The problem is discretized directly by the MFS and an ill-conditioned matrix equation is obtained. A numerical solution of the ill-conditioned equation is unstable. The singular value decomposition (SVD) can give an acceptable solution to such an ill-conditioned matrix equation. The SVD was successfully applied to the MFS for solving a direct problem [12]. Even though we apply the SVD, we still cannot obtain an acceptable solution for the case of the noisy Cauchy data. We use the Tikhonov regularization to obtain a stable regularized solution of the ill-conditioned equation. The regularized solution depends on a regularization parameter. Then, we need to determine a suitable regularization parameter to obtain a better regularized solution. Hansen [3] suggested the L-curve as a method for finding the suitable regularization parameter. It is known that the suitable parameter is the one corresponding to a regularized solution near the "corner" of the L-curve. We can find the corner of the L-curve as a point with the maximum curvature [6].

Under the assumption of uniform distribution of the source and the collocation points, we will numerically indicate that a suitable regularized solution obtained by the L-curve is optimal in the sense that the error is minimized. We will respectively show the accuracy and the optimal regularization parameter against a noise level of the Cauchy data. We will also mention influence of the total numbers of the source and the collocation points on accuracy. We will show that our method is effective for a problem whose solution has singular points outside the computational domain. No multiple-precision arithmetic is required to obtain a good solution. It is noteworthy that such kind of problems can also successfully be solved.

Section 2 introduces the Cauchy problem. In Section 3, the MFS discretizes the problem. In Section 4, the singular value decomposition, the Tikhonov regularization and the L-curve are used to obtain a suitable regularized solution. In Section 5, numerical experiments confirm that the suitable regularization parameter by the L-curve coincides with the optimal one that minimizes the error between the regularized solution and the exact one. The error and the optimal regularization parameter against the noise level of the Cauchy data are respectively shown. Then, our interest is how to choose the following three parameters used in MFS: the numbers of collocation points, the number of source points, and the radius of a circle where source points are distributed. A better choice of the parameters is also observed. A problem with an irregular domain and a problem whose solution has singular points are successfully solved, respectively. Section 6 concludes the paper.

2. Problem setting

We consider the Laplace equation $-\Delta u = 0$ in a two-dimensional bounded domain Ω enclosed by the boundary Γ. We prescribe Dirichlet and Neumann boundary conditions simultaneously on a part of the boundary Γ, denoted by Γ_1, as follows:

$$u = f, \quad \frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma_1,$$

where f and g denote given continuous functions defined on Γ_1, and n the unit outward normal to Γ_1. Then, we need to find the boundary value u on the rest of the boundary $\Gamma_2 := \Gamma \setminus \Gamma_1$ or the potential u in the domain Ω. This problem is called the Cauchy problem of the Laplace equation, and the boundary data are called the Cauchy data.

Our Cauchy problem is described as follows:

Problem 1. For the given Cauchy data $f, g \in C(\Gamma_1)$, find $u \in C(\Gamma_2)$ or $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

$$-\Delta u = 0 \text{ in } \Omega, \tag{1}$$

$$u = f, \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma_1. \tag{2}$$

The Cauchy problem is a well known ill-posed problem. We can show the instability of the solution to the Cauchy problem of the Laplace equation as follows: for example, in the case where

$$\Omega = (0,1)^2 = \{(x,y) : 0 < x < 1, 0 < y < 1\},$$
$$\Gamma = [0,1] \times \{0\} = \{(x,0) : 0 \leqslant x \leqslant 1\},$$
$$f(x,0) = \frac{1}{n^k} \sin(nx), \quad g(x,0) = 0 \quad (k > 0),$$

the solution is given by