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An Asymptotic-Preserving all-speed scheme for the Euler and Navier–Stokes equations

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ABSTRACT

We present an Asymptotic-Preserving ‘all-speed’ scheme for the simulation of compressible flows valid at all Mach-numbers ranging from very small to order unity. The scheme is based on a semi-implicit discretization which treats the acoustic part implicitly and the convective and diffusive parts explicitly. This discretization, which is the key to the Asymptotic-Preserving property, provides a consistent approximation of both the hyperbolic compressible regime and the elliptic incompressible regime. The divergence-free condition on the velocity in the incompressible regime is respected, and an the pressure is computed via an elliptic equation resulting from a suitable combination of the momentum and energy equations. The implicit treatment of the acoustic part allows the time-step to be independent of the Mach number. The scheme is conservative and applies to steady or unsteady flows and to general equations of state. One and two-dimensional numerical results provide a validation of the Asymptotic-Preserving ‘all-speed’ properties.

1. Introduction

The numerical simulation of fluid flows at all Mach numbers is an active field of research. The occurrence of low Mach number regions in a globally compressible flow may be caused by the boundary or initial conditions (e.g. in a fluid at rest subject to a supersonic jet), by the geometry of the problem (e.g. in a nozzle with a large variation of the section), or by the underlying Physics (e.g. in the case of phase changes). This occurrence gives rise to specific numerical issues which are discussed below.

When the Mach number tends to zero, compressible flow equations converge to incompressible equations: the compressible Euler equations in the inviscid case (respectively the compressible Navier–Stokes equations in the viscous case) converge to the incompressible Euler equations (respectively incompressible Navier–Stokes equations). This convergence has been studied mathematically by Klainerman and Majda [32,33] (see also [12,21,45,56] for reviews and references). However, in numerical simulations, it is very difficult to shift from compressible flow equations to incompressible ones in the regions where the Mach-number becomes very small. Therefore, it is necessary to design numerical methods for compressible flows that can handle both the compressible regime (i.e. local Mach-number of order unity) and the incompressible one (i.e. very small local Mach-number). This is the purpose of *‘all-speed schemes’*.

In this work, we derive an All-Speed scheme using the Asymptotic-Preserving methodology. The Asymptotic-Preserving (AP) property is defined as follows. Consider a continuous physical model $\mathcal{M}^\varepsilon$ which involves a perturbation parameter ε

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(here, ε is the scaled Mach-number and $\mathcal{M}^\varepsilon$ represents the compressible Euler or Navier–Stokes model) which can range from $\varepsilon = \mathcal{O}(1)$ to $\varepsilon \ll 1$ values. Let $\mathcal{M}^0$ the limit of $\mathcal{M}^\varepsilon$ when $\varepsilon \to 0$ (here $\mathcal{M}^0$ is the incompressible Euler or Navier–Stokes model). Let now $\mathcal{M}^\varepsilon_\Delta$ be a numerical scheme which provides a consistent discretization of $\mathcal{M}^\varepsilon$ with discrete time and space steps $(\Delta t, \Delta x) = \Delta$. The scheme $\mathcal{M}^\varepsilon_\Delta$ is said to be *Asymptotic-Preserving* (*AP*) if its stability condition is independent of ε and if its limit $\mathcal{M}^0_\Delta$ as $\varepsilon \to 0$ provides a consistent discretization of the continuous limit model $\mathcal{M}^0$. The AP property is illustrated by the commutative diagram of Fig. 1.

The present scheme is derived following the AP methodology and targets the situation of mixed flows where part of the flow has local Mach-number of order unity and is in a compressible regime and part of the flow has very small local Mach-number and is in the incompressible regime. More precisely, our scheme meets the following requirements. It is AP, i.e. it is consistent with both the compressible and incompressible regimes. The divergence-free condition on the velocity in the incompressible regime is explicitly satisfied up to the order of the approximation. The CFL condition is independent of the Mach-number. Therefore, the time-step is not constrained to be inversely proportional to the sound speed like. We remind that classical explicit schemes require such a time-step constraint which is very detrimental to the scheme efficiency in the small Mach-number regime. The scheme is conservative and preserves the correct shock speeds in the compressible regime. At last, the scheme applies to a general equation of state and to steady as well as unsteady flows.

The present work is the continuation of earlier work on the construction of Asymptotic-Preserving schemes for fluid equations in the small Mach-number limit. In [18], a first-order AP scheme is derived for the isentropic Euler equations. A second order version of this scheme based on the Kurganov–Tadmor central scheme methodology is proposed in [58]. Here, we extend the work of [18] to the full Euler and Navier–Stokes equations, i.e. including an energy equations instead of the isentropic assumption. This addition involves more than a simple technical adaptation. Indeed, the scheme has to be strongly modified in the choice of the terms that require an implicit treatment. Some of these terms have to be shifted from the mass to the energy conservation equation. With the use of a real gas equation of state, the resulting pressure equation becomes nonlinear and requires a specific treatment. We also provide a second-order extension of the method based on the classical MUSCL methodology which can apply to a larger software framework than the central scheme methodology. The numerical results will show that the passage to second order is qualitatively necessary to achieve a good accuracy. We also mention [27] which relates to [18] but provides an alternate way of reaching the AP-property.

Understanding why compressible flow solvers perform so poorly in the low Mach-number regime has triggered a vast literature since the seminal work of Chorin [9]. Volpe [63] observed that the numerical error increases when the Mach-number is decreased, at a constant mesh and that the convergence rate deteriorates noticeably. Guillard and Viozat [26] observe that an upwind space discretization leads to pressure fluctuations of the order of the Mach number ε while in the continuous case the pressure fluctuations are of order ε^2. This difference originates from the upwinding terms, and more precisely from the eigenvalues of the Jacobian matrix whose order of magnitude is the sound velocity. The argument has been developed further in [19].

Compressible codes also require an increasingly large computational time as the incompressible regime gets closer. Indeed, the CFL stability condition for an explicit scheme reads $\Delta t \leqslant \frac{\Delta x}{|\lambda^{\max}|}$, where Δt is the time-step, Δx the space step, and $\lambda^{\max}$ is the fastest characteristic wave and can be written $\lambda^{\max} = u \pm c$, u being the fluid velocity and c the sound velocity. In scaled variables (see below for details on the scaling), the Mach number ε appears explicitly in the stability condition as follows:

$$\Delta \tilde{t} \leqslant \frac{\Delta \tilde{x}}{|\tilde{\lambda}^{\max}|} = \frac{\Delta \tilde{x}}{\max |\tilde{u} \pm \frac{\tilde{c}}{\varepsilon}|} = \varepsilon \frac{\Delta \tilde{x}}{\max |\varepsilon \tilde{u} \pm \tilde{c}|}, \tag{1.1}$$

where the tildes denote scaled quantities and the sound speed is now written $\tilde{c}/\varepsilon$ where $\tilde{c} = \mathcal{O}(1)$. The time-step is therefore roughly proportional to the Mach number ε and is dramatically reduced when ε is small.

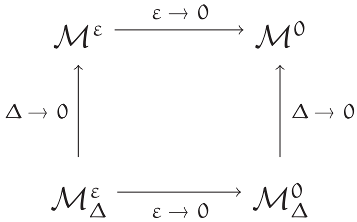

Fig. 1. Asymptotic-Preserving (AP) property: the upper horizontal arrow translates the assumption that the continuous model $\mathcal{M}^\varepsilon$ tends to the limit model $\mathcal{M}^0$ when $\varepsilon \to 0$. The left vertical arrow expresses that $\mathcal{M}^\varepsilon_\Delta$ is a consistent discretization of $\mathcal{M}^\varepsilon$ when the discretization parameter $\Delta \to 0$. The lower horizontal arrow indicates that the scheme $\mathcal{M}^\varepsilon_\Delta$ has a limit $\mathcal{M}^0_\Delta$ when $\varepsilon \to 0$ for fixed Δ. Finally, the right vertical arrow expresses the AP-property: it says that the limit scheme $\mathcal{M}^0_\Delta$ is a consistent discretization of the limit model $\mathcal{M}^0$ when $\Delta \to 0$.