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A study of moving mesh PDE methods for numerical simulation of blowup in reaction diffusion equations

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Abstract

A new concept called the dominance of equidistribution is introduced for analyzing moving mesh partial differential equations for numerical simulation of blowup in reaction diffusion equations. Theoretical and numerical results show that a moving mesh method works successfully when the employed moving mesh equation has the dominance of equidistribution. The property can be verified using dimensional analysis. In several aspects the current work generalizes previous work where a moving mesh equation is shown to have this dominance of equidistribution if it preserves the scaling invariance of the underlying physical partial differential equation and uses a small, constant value for τ (a parameter used for adjusting response time of the mesh movement to the change in the physical solution). Also, cases with both constant and variable τ are considered here.

MSC: 65N50; 65M50; 35K55; 35K57; 65N35

Keywords: Moving mesh; Blowup; Reaction diffusion equations; Mesh adaptation

1. Introduction

We are concerned with the numerical solution of reaction diffusion equations whose solutions become unbounded (or blowup) in a finite time. This type of partial differential equation (PDE) arises from mathematical idealizations of models describing combustion in chemicals or chemotaxis in cellular aggregates, the formation of shocks in the inviscid Burgers' equation, and the space-charge equations; e.g. see Pao [18]. Such a blowup in the solution often represents an important change in the properties of the model, such as the ignition of a heated gas mixture, and it is important that it is reproduced accurately by a numerical computation.

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Since a blowup typically occurs on increasingly small length scales as well as time scales, it is essential to use an adaptive mesh in the numerical simulation. Two types of mesh adaptation have been commonly used, mesh refinement [4] and mesh movement [8]. With the former approach, mesh points are added as the length scale is getting smaller whereas in the latter approach a fixed number of mesh points are moved to resolve the increasingly small length scale.

In this paper we are interested in the moving mesh solution of blowup problems and focus particularly on the MMPDE (moving mesh PDE) moving mesh method developed in [13]. It has been shown in [8] that the key to the success of the method is to have MMPDEs preserve the scaling invariance of the underlying physical equation. This idea has since been used with success in most computations of blowup solutions that use the MMPDE method; e.g. see [7]. However, as we shall see in Sections 4 and 5, preserving the scaling invariance is neither sufficient nor necessary in general for MMPDEs to work satisfactorily, although it is sufficient for the particular approach considered in [8] where the parameter τ used in MMPDEs for adjusting the response time of mesh movement to the change in the physical solution is taken to be constant. Approaches with variable τ have also been used successfully by a number of researchers; e.g. see [12,19]. Thus, from both the theoretical and practical points of view there is a need for an in-depth study of the moving mesh method for blowup problems.

The objective of this paper is to present such a study. We are most concerned with conditions under which MMPDEs work satisfactorily. The tool we use is a new concept called *the dominance of equidistribution*: the terms representing the well known equidistribution principle [10,11] for mesh adaptation dominate other terms in the equation. We show that the solution of an MMPDE stays closely to the solution of the equidistribution principle when it has this property, implying that the dominance of equidistribution is sufficient for an MMPDE to work satisfactorily. Moreover, we show that the dominance of equidistribution can often be straightforwardly verified using dimensional analysis. A special case of the dominance of equidistribution is to have an MMPDE preserve the scaling invariance of the underlying physical PDE and to choose a small τ – the approach used in [8]. Furthermore, the concept applies to general situations, including those with constant and variable τ, and even in multi-dimensions.

It is worth mentioning some history of numerical simulation of blowup. The first works on the topic are Nakagawa [16] and Nakagawa and Ushijima [17] where finite difference and finite element schemes on a uniform mesh are employed and analyzed for blowup for PDE (1) with $p = 2$. A mesh refinement strategy is proposed by Berger and Kohn [4] and a moving mesh method is presented by Budd et al. [8] for the numerical solution of blowup problems. A survey is given by Bandle and Brunner [2]. Recent works include [1,5–7,9].

An outline of the paper is as follows. The MMPDE method is described in the next section for a classic problem with blowup solutions. The dimensional analysis for both the physical and mesh equations is presented in Section 3. The question of how to verify the dominance of equidistribution using dimensional analysis is also addressed in this section. Theoretical and numerical analyses of MMPDEs with constant and solution-dependent τ are given in Sections 4 and 5, respectively. Additional comments and conclusions are contained in the final section.

2. Moving mesh PDE method

We study the moving mesh method for a classic problem with a blowup solution:

$$u_t = u_{xx} + u^p, \quad p > 1 \tag{1}$$

subject to the boundary and initial conditions

$$u(0,t) = u(1,t) = 0, \tag{2}$$

$$u(x,0) = u_0(x) > 0. \tag{3}$$

It is known that when the initial solution is sufficiently large, the solution of the initial-boundary value problem tends to infinity at a point, say $x^* \in (0,1)$, as $t \to T$ for some finite time $T > 0$, x^* and T are referred to as the blowup point and time, respectively. A more precise description of the blowup profile of the solution is given in the following theorem (see [3] and references therein):