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A class of discontinuous Petrov–Galerkin methods. Part IV: The optimal test norm and time-harmonic wave propagation in 1D

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ABSTRACT

The phase error, or the pollution effect in the finite element solution of wave propagation problems, is a well known phenomenon that must be confronted when solving problems in the high-frequency range. This paper presents a new method with *no phase errors* for one-dimensional (1D) time-harmonic wave propagation problems using new ideas that hold promise for the multidimensional case. The method is constructed within the framework of the discontinuous Petrov–Galerkin (DPG) method with optimal test functions. We have previously shown that such methods select solutions that are the best possible approximations in an energy norm dual to any selected test space norm. In this paper, we advance by asking what is the *optimal test space norm* that achieves error reduction in a given energy norm. This is answered in the specific case of the Helmholtz equation with L^2-norm as the energy norm. We obtain uniform stability with respect to the wave number. We illustrate the method with a number of 1D numerical experiments, using discontinuous piecewise polynomial *hp* spaces for the trial space and its corresponding optimal test functions computed approximately and locally. A 1D theoretical stability analysis is also developed.

1. Introduction

The aim of this paper is to introduce a new methodology to design schemes for wave-propagation problems. It is a continuation of our research on discontinuous Petrov–Galerkin (DPG) methods [10–12]. Our previous papers applied the DPG methodology to get new methods for convective and diffusive phenomena. In this paper, we apply it to wave propagation after developing additionally needed theoretical tools.

The numerical solution of wave propagation problems at high frequencies has been recognized as an outstanding challenge in numerical analysis. In general, numerical methods for wave propagation are subject to the effect of *pollution*: increasing the frequency, while maintaining the approximation quality of the numerical discretization, results in a divergence of the computed result from the best approximation the discretization is capable of. In the context of finite element methods, the pollution error may be characterized as follows [24]: given that the exact solution u lies in a space U normed by $\|\cdot\|_U$, and the discrete solution u_h in an approximation subspace $U_h \subset U$, one observes that:

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$$\frac{\|u-u_h\|_U}{\|u\|_U} \leqslant C(k) \inf_{w_h \in U_h} \frac{\|u-w_h\|_U}{\|u\|_U},$$

where

$$C(k) = C_1 + C_2 k^{\beta} (kh)^{\gamma},$$

with k being the wavenumber, and h being the element size. The infimum measures the best approximation error. This is typically small when kh is small, i.e., when enough elements per wavelength are used. Additional kh dependence may arise through γ. However, more troublesome is the k-dependence in $C(k)$ measured by β. It reflects the growing instability of the problem even before discretization, i.e. the inf–sup constant decreases as k increases. Generally, the exponent β is found to be one [24,26] – in other words, the "pollution" term in the error increases linearly with frequency. For many model problems, the pollution is manifested as a phase error which accumulates over the domain, and the concepts of pollution, phase error, and discrete wavenumbers are therefore all closely related. The growth of the pollution error, combined with the already difficult problem of approximating the highly oscillatory solutions of wave problems, can render the numerical solution extremely expensive for high wavenumbers.

The main result of our application of the DPG methodology to one-dimensional wave propagation is a Petrov–Galerkin method which is free of pollution, i.e. $\beta = 0$. Additionally, our method also has $\gamma = 0$. A number of previous methods have achieved zero β in 1D, while reducing the severity of the pollution error in higher dimensions. One can find surveys of such methods in, e.g., [28,20]. Broadly, they may be classified as follows: Galerkin/least-squares based methods [21,29], which achieve improved stability by adding least squares residual terms to the standard Galerkin sesquilinear form; methods utilizing specialized, under-integrating quadrature rules [1] which reduce the phase error, as indicated by dispersion analysis of an interior stencil; and methods incorporating exact solutions of the Helmholtz equation (in particular, plane waves) within the trial space basis [3,15–17,22].

Petrov–Galerkin (PG) formulations also appear frequently in the construction of stabilized methods (see, e.g., [13,14,23]). Common to such methods is the introduction of local problems which are solved to provide a trial/test space pair which provides enhanced stability. A few of these methods have attempted to address in particular the Helmholtz equation.

In the nearly optimal Petrov–Galerkin method (NOPG) of Barbone and Harari [5], the authors construct a method with the goal of achieving the best approximation in the H^1 semi-norm in a given trial space. They show that the corresponding minimization problem leads to a Petrov–Galerkin formulation with optimal test functions with global support. Then, by considering only local test functions constructed by adding bubbles to the standard basis functions, they arrive at a more practical formulation which approximates the H^1-optimal result. For rectangular/hexahedral elements, the bubble functions may be determined analytically; more generally, the bubbles may be approximated numerically through local Galerkin problems. In certain cases, the method is equivalent to that of residual-free bubbles [18].

The quasi-optimal Petrov–Galerkin (QOPG) method of Loula and Fernandes [25] considers test functions constructed from a linear combination of standard bilinear Lagrangian basis functions and additional bubbles which are products of the same basis functions. The test functions are determined by solving locally a least-squares problem attempting to minimize a residual corresponding to the Lagrange interpolant of plane waves of all directions. For a uniform mesh, the phase error determined by analysis of an interior stencil is of the same order as that of the quasi-stabilized FEM (QSFEM) of Babuska et al. [2], i.e.,

$$\frac{|k-k_h|}{k} \leqslant 1.5 \left(\frac{(kh)^6}{774144} \right), \tag{1.1}$$

where k_h is a "discrete" wave number.

In general, both (NOPG and QOPG) methods require simple preprocessing techniques which can be implemented in existing FEM codes with little extra computational cost. However, both methods fit within the class of generalized finite elements methods (GFEM) analyzed in [4] when restricted to structured meshes. Therefore, we know that in 2D they perform (in the best case) with the same order of phase error as the optimal result [2], i.e., the expression (1.1).

The method we present for Helmholtz problems is very similar in spirit to these other approaches, i.e., it attempts to achieve optimal results in some sense by local computation of corresponding optimal test functions. The use of the DPG setting is where we depart. We have developed such formulations together with the concept of optimal test functions in [10–12] for convective problems (DPG variational formulations were also considered in [6], but their objective was not to find the best possible test space). Rather than starting from a traditional H^1 variational formulation in terms of pressure, the DPG setting introduces a mixed formulation for both pressure and velocity, which are now in L^2, as well as additional fluxes. We then aim for test functions that yield the best trial approximations in the L^2 norm for both pressure and velocity. The mixed formulation and the discontinuity of the functional spaces is needed to derive an easy, practical, and inexpensive way to compute the optimal test space. Compared to other PG approaches (e.g., [5] or [25]), the method may be difficult to implement within existing classical FEM codes, but fits perfectly within the framework of hybrid methods like the original DPG method developed in [6]. The essential difference is in the computation of optimal test functions, an operation performed purely on the element level using a simple preprocessing routine. Additionally, for a low price, our method also obtains local error indicators for an *hp*-adaptive algorithm (see [12]).

The crucial property of the DPG methodology is that it guarantees the *best approximation property* in the so-called energy (dual or residual) norm [11]. This norm is problem-dependent – it is implied by the operator governing the problem *and the*