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The finite volume scheme preserving extremum principle for diffusion equations on polygonal meshes

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ABSTRACT

We construct a new nonlinear finite volume scheme for diffusion equation on polygonal meshes and prove that the scheme satisfies the discrete extremum principle. Our scheme is locally conservative and has only cell-centered unknowns. Numerical results are presented to show how our scheme works for preserving discrete extremum principle and positivity on various distorted meshes.

1. Introduction

Accurate and reliable discretization methods are very important for the numerical simulations of Lagrangian hydrodynamics. Development of a new discrete scheme for diffusion equation should satisfy some desirable properties [18,36], specifically it is better to satisfy the following requirements:

– locally conservative;
– monotone, or more generally satisfy the discrete extremum principle;
– reliable on unstructured anisotropic meshes that may be severely distorted;
– allow heterogeneous full diffusion tensors;
– result in a sparse system with minimal number of non-zero entries;
– have the accuracy that is higher than the first order for smooth solutions;
– have only cell-centered unknowns;
– ensure the continuity of the normal flux through cell interfaces.

For diffusion problems the so-called extremum principle includes maximum principle and minimum principle. Usually, the monotonicity refers to be nonnegativity maintaining, which is a special case of the minimum principle. In the setting of linear diffusion problems the extremum property is equivalent with the monotonicity. The discrete extremum principle is the discrete version of the well-known extremum principle for diffusion equations. In [25] it is pointed out that a scheme violating extremum principle can lead to two problems: (a) fully implicit discretization with large time-steps has relatively

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poor accuracy and (b) spurious negative values are generated. Moreover in [25] it is proved that a linear operator, resulting from the discretization of diffusion equations, satisfies extremum principle if and only if it is both differential and nonnegativity maintaining. Since nonlinear discrete schemes are involved in our discussion, we will distinguish maximum principle and minimum principle. For general cases (including nonlinear cases), the discrete extremum principle is more restrictive than monotonicity.

Monotonicity is one of the key requirements to discretization schemes. In the context of anisotropic thermal conduction, the scheme without preserving monotonicity can lead to the violation of the entropy constraints of the second law of thermodynamics, causing heat to flow from regions of lower temperature to higher temperature. In regions of large temperature variations, this can cause the temperature to become negative. In order to avoid negative temperature, the scheme must be monotone. The extremum principle is also one of the key requirements to discretization schemes. The discrete extremum principle can ensure that there is no spurious oscillations for the numerical solution. From the extremum principle, we know that the maximum and minimum of solution for a diffusion problem without source and sink in the interior of physical domain can only be attained at the boundary of physical domain.

It is well known that classical finite volume and finite element schemes fail to satisfy the discrete extremum principle for strong anisotropic diffusion tensors and on distorted meshes [10,16,24]. To our knowledge, a linear scheme satisfying all the above requirements is not known. There are many linear schemes satisfying part of requirements mentioned above, but not all of them. For example, the schemes in [9,34] only consider the case of diffusion coefficient being scalar, the schemes in [17,22,23] are monotone under certain severe (geometric) restrictions, and the monotonicity criteria for the schemes in [1,2,6,7,11,29,30] have not been analyzed. The schemes in [13–15,32,33,35] have face-centered unknowns or vertex unknowns in addition to the cell-centered unknowns, and the monotonicity criteria for them have not been studied yet.

Some algorithms in [28] based on slope limiters are proposed to preserve the monotonicity. In [21], based on repair technique and constrained optimization, two approaches have been suggested to enforce discrete extremum principle for linear finite element solutions of general elliptic equations with self-adjoint operator on triangular meshes. A finite volume scheme for diffusion problems on general meshes applying monotony constraints is presented and analyzed in [3].

The criteria for the monotonicity of control volume methods on quadrilateral meshes was derived in [24], and it was shown that it is impossible to construct linear nine-point methods which unconditionally satisfy the monotonicity criteria when the discretization satisfies local conservation and exact reproduction of linear solution.

On the other hand, a few nonlinear schemes [8,18,26] have been proposed to guarantee monotonicity. A nonlinear stabilized Galerkin approximation of the Laplace operator has been analyzed in [8] and a nonlinear finite volume scheme for highly anisotropic diffusion operators on unstructured triangular meshes has been proposed in [26]. It was shown in [26] that the scheme is monotone only for parabolic equations and sufficiently small time steps. The nonlinear finite volume scheme suggested in [26] has been further developed and analyzed for elliptic problems in [18], which satisfies the above requirements on triangular meshes. They proved that the scheme is monotone on triangular meshes for strongly anisotropic and heterogeneous full tensor coefficients. An interpolation-free nonlinear monotone scheme is presented in [19], and it has been extended to the advection diffusion equations on unstructured polygonal meshes in [20]. A nonlinear finite volume scheme satisfying extremum principle for diffusion operators on triangular cells is presented in [27]. In [5] a nonlinear diamond scheme on unstructured meshes of d-simplices (triangle for $d = 2$ and tetrahedra for $d = 3$) for convection–diffusion problems is proposed, in which the face gradient required to discretize the diffusive fluxes at any internal face is reformulated as a nonlinear average of the one-side gradients by suitably designing solution-dependent weights. This paper is a nontrivial generalization of the nonlinear strategy to diffusion schemes on any star-shaped polygonal meshes.

In [36], the authors construct a nonlinear monotone finite volume scheme for linear elliptic problems and parabolic problems with anisotropic and heterogeneous full tensor coefficients, based on an adaptive approach of choosing stencil in the construction of discrete normal flux. Moreover, the authors proved that the scheme is monotone on any star-shaped polygonal meshes, without any special restriction for the choice of cell centers. Numerical results show that the scheme obtains almost second order convergence. This monotone scheme has been extended to the nonequilibrium radiation diffusion equations in [31]. However, we do not know whether these schemes satisfy the discrete extremum principle.

In this paper we will further develop the nonlinear finite volume schemes, and following the idea in [36] we will construct a new nonlinear finite volume scheme which satisfies the discrete extremum principle. For our scheme, we can simply take collocation points as the cell-centers which are defined in Lagrangian hydrodynamics algorithm for polygonal meshes. It follows that our scheme avoids a remap from the values on collocation points to those on cell-centers, and would be suitable for coupled radiation diffusion/hydrodynamics calculations on such meshes. The consistency and coercivity of our scheme (including the consistency of the discrete fluxes) are very important issues on theoretical analysis such as the stability and convergence of the scheme, and will be discussed in future work. It should be pointed out that the theoretical results for the discrete schemes of diffusion equation on meshes with certain geometric assumptions are discussed in some references (see [12] and references therein, [4], etc.), but it is not easy to be generalized to the finite volume schemes of diffusion equation with discontinuous coefficient on general meshes.

The remainder of this article is organized as follows. In Section 2 we describe the problem and give some notations. In Section 3, we describe the construction of the nonlinear finite volume scheme. In Section 4, we prove that this scheme satisfies the discrete extremum principle. Then in Section 5 we present some numerical results to illustrate the features of the scheme. Finally we give some conclusions in Section 6.