Journal of Computational Physics 230 (2011) 2620–2642

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Analysis of the monotonicity conditions in the mimetic finite difference method for elliptic problems ☆

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ARTICLE INFO

Article history:
Received 2 September 2010
Received in revised form 22 December 2010
Accepted 23 December 2010
Available online 14 January 2011

Keywords:
Mimetic finite differences
Discrete maximum principle
Monotone matrix
M-matrix
Monotone scheme
Mesh refinement

ABSTRACT

The maximum principle is one of the most important properties of solutions of partial differential equations. Its numerical analog, the discrete maximum principle (DMP), is one of the most difficult properties to achieve in numerical methods, especially when the computational mesh is distorted to adapt and conform to the physical domain or the problem coefficients are highly heterogeneous and anisotropic. Violation of the DMP may lead to numerical instabilities such as oscillations and to unphysical solutions such as heat flow from a cold material to a hot one. In this work, we investigate sufficient conditions to ensure the monotonicity of the mimetic finite difference (MFD) method on two- and three-dimensional meshes. These conditions result in a set of general inequalities for the elements of the mass matrix of every mesh element. Efficient solutions are devised for meshes consisting of simplexes, parallelograms and parallelepipeds, and orthogonal locally refined elements as those used in the AMR methodology. On simplicial meshes, it turns out that the MFD method coincides with the mixed-hybrid finite element methods based on the low-order Raviart–Thomas vector space. Thus, in this case we recover the well-established conventional angle conditions of such approximations. Instead, in the other cases a suitable design of the MFD method allows us to formulate a monotone discretization for which the existence of a DMP can be theoretically proved. Moreover, on meshes of parallelograms we establish a connection with a similar monotonicity condition proposed for the Multi-Point Flux Approximation (MPFA) methods. Numerical experiments confirm the effectiveness of the considered monotonicity conditions.

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1. Introduction

The existence of a maximum principle is a fundamental property of the solutions of the elliptic problems [29,31]. Let p be the solution of the elliptic problem $L(p) = f$ posed in an open domain Ω, where $L(\cdot)$ is a general second-order elliptic operator and f the source term. Suppose that the source term is nonnegative, $f \geqslant 0$, and that the coefficients of L are regular enough. Then p has no minimum in Ω. More precisely, if there exists a point $\mathbf{x}_0 \in \Omega$ such that $p(\mathbf{x}_0) \geqslant p(\mathbf{x})$ for all other $\mathbf{x} \in \Omega$, then p is constant in Ω. This version of the maximum principle is known as *Hopf's lemma* [31] and proved under the condition that the diffusion tensor is continuously differentiable.

☆ This work was carried out under the auspices of the National Nuclear Security Administration of the US Department of Energy at Los Alamos National Laboratory under Contract No. DE-AC52-06NA25396 and the DOE Office of Science Advanced Scientific Computing Research (ASCR) Program in Applied Mathematics Research.

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0021-9991/$ - see front matter Published by Elsevier Inc.
doi:10.1016/j.jcp.2010.12.039

The possibility of reproducing this property for the approximation of an elliptic problem, thus ensuring a *discrete maximum principle* (DMP) for the numerical solution, is of major importance in the development of numerical methods that are robust and accurate. As pointed out in [46], this property may be crucial for the numerical solution of complex engineering applications, e.g., multiphase flow problems in heterogeneous porous media with anisotropic diffusivity. In a multiphase flow some variables (e.g., saturations) are solutions of hyperbolic equations, while the pressure is the solution of an elliptic equation. Phase transitions are pressure dependent and may be seriously affected by spurious effects like numerical oscillations in the pressure field. For example, if the approximate value of the pressure lies below the bubble-point curve of the mixture, while the actual pressure lies above it, artificial gas may be liberated, yielding a numerical solution that is strongly divergent with the true solution of the model.

A huge amount of works in the literature concerning finite volumes and finite elements of linear and nonlinear parabolic and elliptic partial differential equations investigate the issue of reproducing at the discrete level this fundamental property of the exact solutions. Among the most recent work, we mention [16,19,28,35,36,52]. It turns out that classical finite volume and finite element schemes may fail to satisfy a DMP for strong anisotropic diffusion tensors and/or distorted meshes [27,46]. For example, in [27,46], monotonicity of the numerical scheme can be achieved only for a scalar diffusion tensor under some restrictive condition on the shape of the parallelograms and for a specific range of the diffusion value. A different class of finite volume schemes that are able to preserve the maximum principle is based on a nonlinear discretization [11,42,48]. In a finite element framework, the nonlinear discretization which satisfies a DMP has been proposed in [37]. We also mention that, quite recently, a post-processing technique based on a "repair" concept has been proposed in [43] to recover a positivity condition from a numerical solution that does not satisfy a maximum principle. Finally, analysis of the DMP for higher-order finite element and finite volume methods is limited mainly to 1D problems (see [49,51] and references there in).

Different formulations of a DMP are possible since they can be derived from different formulations of the continuous maximum principle [34]. One example is the study in [51], where the method is said to satisfy a maximum principle for problems with zero source term, if p on Ω is less than or equal to p on $\partial\Omega$. Another formulation of a DMP follows from requiring the nonnegativity of the inverse of the stiffness system. Suppose that the discretization of $L(p) = q$ with homogeneous Dirichlet boundary conditions leads to the system of discrete equations $Ap^h = q^h$. If $A^{-1} \geqslant 0$, i.e. if each element of A^{-1} is nonnegative, then the discrete system satisfies a property formally similar to that of the continuous system: $q^h \geqslant 0$ implies that $p^h \geqslant 0$.

A nonsingular matrix A whose inverse has the sign property discussed above is called *monotone* [50]. In this paper as well as in [46] a numerical method is called monotone if it leads to a monotone matrix. An effective way to ensure that the monotonicity property holds is to construct a numerical method such that the matrix A is an M-matrix [10]. In fact, a great number of spatial finite difference discretizations for second-order elliptic problems yield M-matrices [12–15,24–26], which guarantees the monotonicity of the numerical method. For discretizations not leading to M-matrices, far less is known. An early analysis was conducted by Bramble and Hubbard [13,15]. In [51], a weaker form of the maximum principle was analyzed for more general problems. In [46] some local criteria were recently investigated to ensure monotone solutions of elliptic problem with scalar diffusion tensors using a nine-point scheme on meshes of regular parallelograms.

In this paper, we consider discrete formulations of the maximum principle and derive sufficient monotonicity criteria for the Mimetic Finite Difference (MFD) method in mixed or mixed-hybrid form [17]. The MFD methods mimic important properties of physical and mathematical models such as fundamental identities of the tensor and vector calculus, conservation laws, solution symmetry, and positivity. The MFD method and its earlier version, the support-operator method, has been successfully employed for solving problems of continuum mechanics [44], electromagnetism [32,38], gas dynamics [20], linear diffusion (see, e.g., [6,9,30,33,39,45], and references therein), convection–diffusion [23], Stokes [5,4,7], elasticity [3], eigenvalues [21] and two-phase flows in porous media [1,40]. A posteriori estimators have also been developed in [2,8].

The monotonicity criteria that we develop are local as they are imposed on *positivity* and *sparsity* of the mimetic inner product at the elemental level. Using such conditions we construct monotone MFD methods for general (also anisotropic) diffusion tensors and different families of meshes that are widely in use in the scientific community. For instance, parallelograms in 2D and oblique parallelepipeds in 3D are used to represent tilted layers, while meshes from Adaptive Mesh Refinement (AMR) techniques [47] are used to increase local accuracy of the numerical solution. Concerning accuracy and stability, our criteria are not restrictive since all theoretical results about the convergence of the numerical flux and the superconvergence of the pressure variable derived in [17] still hold. It turns out that the resulting matrix for the Lagrange multipliers of the mixed-hybrid form is a nonsingular M-matrix. This fact ensures that the inverse matrix has only nonnegative elements.

The outline of this article is as follows. In Section 2, we describe the problem and recall fundamental results on the maximum principle. In Section 3, we study the algebraic monotonicity conditions for mixed-hybrid discretizations. In Section 4, we formulate sufficient monotonicity conditions for the MFD method. We study separately simplicial meshes, quadrilateral meshes, hexahedral meshes and orthogonal AMR meshes. In Section 5, we illustrate our findings with numerical results. In Section 6, we offer the final remarks and conclusions.

2. Discretization of the diffusion problem in mixed form

Let Ω be a bounded, simply connected, open subset of R^d for $d = 2, 3$ with boundary Γ. For simplicity, we assume that Ω be either a polyhedral domain for $d = 3$ or a polygonal domain for $d = 2$. We consider the diffusion of a scalar quantity p in an anisotropic heterogeneous medium filling Ω, which is governed by the second-order elliptic partial differential equation