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Short Note

A primal formulation for the Helmholtz decomposition

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Abstract

In 1999, Jean-Paul Caltagirone and Jérôme Breil have developed in their paper [Caltagirone, J. Breil, Sur une méthode de projection vectorielle pour la résolution des équations de Navier–Stokes, C.R. Acad. Sci. Paris 327(Série II b) (1999) 1179–1184] a new method to compute a divergence-free velocity. They have used the **grad**(div) operator to extract the solenoidal part of a given vector field. In this contribution we explain how this method can be considered as a real Helmholtz decomposition and we present a stable approximation in the framework of spectral methods. Numerical results are presented to illustrate the efficiency of this approach.

Keywords: **grad**(div) operator; Stable approximation; Helmholtz decomposition

1. Introduction

The approximation of the **grad**(div) operator pervades many applied physics domains. Besides the ideal ocean wave problem without Coriolis force and no friction [15], it arises in the Maxwell equations [11] and in the Navier–Stokes equations for fluid flow problems when using a penalty formulation for the incompressibility condition [14]. The problem also arises in the ideal linear magneto hydrodynamics equations when computing the stability behavior of a fusion plasma device [16]. Another original application of this operator was introduced by J.P. Caltagirone and J. Breil in their paper [13] where they used this operator to extract from a given velocity field its solenoidal part. These authors had christened it *vector projection* which consists in solving the following problem: Let $\mathbf{u}^*$ be a non-divergence free velocity field, find a couple of vector fields $(\mathbf{u},\mathbf{v})$ such that

$$-\boldsymbol{\nabla}(\nabla\cdot\mathbf{u})=\boldsymbol{\nabla}(\nabla\cdot\mathbf{u}^*),\quad \text{in } \Omega, \tag{1.1}$$

$$\mathbf{u}\cdot\mathbf{n}=0,\quad \text{on } \partial\Omega, \tag{1.2}$$

$$\mathbf{v}=\mathbf{u}+\mathbf{u}^*,\quad \text{in } \Omega, \tag{1.3}$$

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where $\mathbf{v}$ and $\mathbf{u}$ are respectively divergence-free and curl-free. Here $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is a simply connected and bounded domain with Lipschitzian border. $\mathbf{n}$ denotes the outer unit normal along the boundary.

The objective of this note is on the one hand to explain how the previous system can be considered as a Helmholtz decomposition step and on the other hand to present a stable discretization in the framework of spectral methods. We end this note by presenting a relevant numerical experiment.

Some notations – The symbol $L^2(\Omega)$ stands for the usual Lebesgue space and $H^1(\Omega)$, the Sobolev space, involves all the functions that are, together with their gradient, in $L^2(\Omega)$. The $\mathscr{C}(\Omega)$ denotes the space of continuous functions defined in Ω.

2. Continuous problems and their variational formulations

In order to write the continuous problem in its variational form we introduce the relevant spaces of functions.

Let $H(\text{div},\Omega)$ denote the space (see [12])

$$H(\text{div}, \Omega) = \{\mathbf{w} \in (L^2(\Omega))^d; \text{div}\,\mathbf{w} \in L^2(\Omega)\},$$

endowed with the natural norm

$$\|\mathbf{w}\|_{H(\text{div},\Omega)} = (\|\mathbf{w}\|^2_{(L^2(\Omega))^d} + \|\text{div}\,\mathbf{w}\|^2_{L^2(\Omega)})^{1/2}.$$

The continuous problem we consider reads: *Find* $\mathbf{u}$ *in* $H(\text{div}, \Omega)$ it such that:

$$-\nabla(\nabla \cdot \mathbf{u}) = \mathbf{f}, \quad \text{in } \Omega, \tag{2.4}$$

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega, \tag{2.5}$$

where $\mathbf{f}$ is a given data.

Since curl $(\mathbf{grad}\,\cdot) \equiv \mathbf{0}$ we notice that a necessary condition for the existence of a solution to problems (2.4) and (2.5) is that curl $\mathbf{f} = \mathbf{0}$ and by consequence we can state the existence of a function $\varphi(x, y)$ such that

$$\mathbf{f} = \mathbf{grad}\,\varphi.$$

This leads to restate the basic problem as: *For a given* $\varphi \in L^2_0(\Omega)$, *find* $\mathbf{u} \in H(\text{div},\Omega)$ *such that*

$$-\nabla(\nabla \cdot \mathbf{u}) = \nabla\varphi, \quad \text{in } \Omega, \tag{2.6}$$

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega, \tag{2.7}$$

where $L^2_0(\Omega)$ denotes the $L^2(\Omega)$ subspace of functions having zero average values. This formulation is equivalent to the dual one that reads: *For a given* $\varphi \in L^2_0(\Omega)$ *find* $\mathbf{u} \in X(\Omega)$ *and* $\psi \in L^2_0(\Omega)$ *such that*:

$$\mathbf{u} - \nabla\psi = 0, \quad \text{in } \Omega, \tag{2.8}$$

$$-\nabla \cdot \mathbf{u} = \varphi, \quad \text{in } \Omega, \tag{2.9}$$

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega, \tag{2.10}$$

where

$$X(\Omega) = \{\mathbf{w} \in H(\text{div}, \Omega); \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

This dual formulation can be rewritten as a classical Helmholtz decomposition, indeed: Let $\mathbf{u}^*$ and $\mathbf{v}$ be two vector fields such that $\nabla \cdot \mathbf{u}^* = \varphi$, $\mathbf{u}^* \cdot \mathbf{n} = 0$ and $\mathbf{v} = \mathbf{u} + \mathbf{u}^*$. The problem (2.8)–(2.10) then becomes: *Find* $\mathbf{v} \in X(\Omega)$ *and* ψ in $L^2_0(\Omega)$ *such that*

$$\mathbf{v} - \nabla\psi = \mathbf{u}^*, \quad \text{in } \Omega, \tag{2.11}$$

$$\nabla \cdot \mathbf{v} = 0, \quad \text{in } \Omega, \tag{2.12}$$

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega. \tag{2.13}$$

Consequently the Helmholtz decomposition of the vector field $\mathbf{u}^*$ can be achieved using either the primal formulation (1.1)–(1.3) or its equivalent dual one (2.11)–(2.13).