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A rescaling scheme with application to the long-time simulation of viscous fingering in a Hele–Shaw cell

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Abstract

In this paper, we present a time and space rescaling scheme for the computation of moving interface problems. The idea is to map time–space such that the interfaces can evolve exponentially fast in the new time scale while the area/volume enclosed by the interface remains unchanged. The rescaling scheme significantly reduces the computation time (especially for slow growth), and enables one to accurately simulate the very long-time dynamics of moving interfaces. We then implement this scheme in a Hele–Shaw problem, examine the dynamics for a number of different injection fluxes, and present the largest and most pronounced viscous fingering simulations to date.

Keywords: Hele–Shaw; Saffman–Taylor instability; Boundary integral method; Moving boundary problems; Fractal; Self-similar

1. Introduction

Many physical problems involve moving interfaces, such as the growth of crystals, the dynamics of Hele–Shaw flows, etc. Characterizing the formation and dynamics of complex interface morphologies due to instability has long been a challenging research topic (e.g. [1–6]). While numerical simulation has become one of the most important tools for investigating the motion of interfaces, it is still difficult to obtain accurate approximations, especially for the long-time evolution of interfaces. Specifically, difficulties arise because one must efficiently and accurately resolve the multiple time and space scales involved in the physics that lead to the development of complex morphologies.

There are many computational methods that have been developed for simulating interfacial instabilities, such as boundary integral methods (e.g. see [7,8,33]), the level set methods (e.g. see [10–13]), volume of fluid methods (e.g. see [14,15]), front-tracking methods (e.g. see [16,17]), immersed interface methods (e.g. see [18,19]), phase field methods (e.g. see [20–23]) as well as hybrid methods (e.g. see [24–27]). These methods are usually coupled with adaptive mesh algorithms to gain more efficiency. When applicable (e.g. for piecewise

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homogeneous problems) boundary integral methods are typically the most accurate methods for simulating interface motion because the dimensionality of the problem is reduced by one and there are well-developed accurate and stable discretizations of boundary integral equations. Further, while boundary integral methods are important in their own right, they can also serve as benchmarks for other more general methods.

A decade ago, Hou, Lowengrub and Shelley (HLS94) significantly advanced the state-of-the-art of boundary integral methods in a study of viscous fingering (the Saffman–Taylor instability) in a Hele–Shaw cell [9], as well as studies of the motion of vortex sheets with surface tension [28]. The algorithm relied on an analysis of the equations at small spatial scales (SSD [9]) that identified and removed the source of stiffness introduced by surface tension. This enabled the use of large time steps and made long-time simulations possible. To further enhance the efficiency of the algorithm, the fast multipole method [41] was used to evaluate the boundary integrals. Later on, this method was successfully adapted to other physical problems such as microstructural evolution in inhomogeneous elastic domains [34], solid tumor growth [35], crystal growth [36–38], etc. The resulting body of research has led to many interesting discoveries (e.g. see [29–32] and the review article [33]).

Very recently, Fast and Shelley [39] re-ran the long-time simulation of a Hele–Shaw bubble originally presented in [9]. By comparing the CPU time with that used in [9], Moore's law was verified: the computation power has increased a hundredfold since 1994. For example, it took only 14 h to reproduce the bubble simulation in [9], roughly 1% of the 50 days required in 1994. Using the same wall time (50 days), Fast and Shelley ran the simulation 10 times longer and computed an interface using up to $N = 32{,}768$ mesh points. The bubble assumes a complex fingering pattern and is about 4 times larger (in radius) than the one presented in [9]. By computing farther in time, Fast and Shelley identified the emergence of a new scaling regime in the relationship between the area $A(t)$ and the arclength $L(t)$, which reflects the highly ramified bubble structure.

Complex viscous fingering patterns reflect the Saffman–Taylor instability [5], which occurs when the stabilizing forces (e.g. surface tension) and the destabilizing driving force (e.g. flux or flow injection rate) are not balanced. For example, in [9,39], a constant flux (constant air injection rate) was used. As suggested by the linear stability analysis, as the bubble grows, larger and larger wavenumbers become unstable, which leads to the nonlinear development of a ramified pattern by repeated tip-splitting. Moreover, for a constant flux, the equivalent bubble radius evolves as $\mathrm{d}R/\mathrm{d}t \sim R^{-1}$, where R is the radius of a circle with the same area as the bubble. Consequently the velocity of the bubble, $\mathrm{d}R/\mathrm{d}t$, decreases as R increases (the bubble grows). From the perspective of numerical computation, this makes the problem highly challenging. Not only does the complex fingering pattern require many mesh points to resolve the interface, but also the intrinsic slow growth (e.g. due to an applied constant injection flux) makes simulations of the evolution to large sized bubbles very expensive.

In this paper, we develop a rescaling scheme which enables one to accurately simulate the long-time dynamics of moving interfaces. In this approach, time is scaled such that the bubble size grows exponentially fast in new time scale, and space is scaled such that the area is constant in the new frame. In the numerical scheme, an analytical formula is used to determine the overall growth due to flux and is therefore free of discretization error. The scheme overcomes the intrinsic slow growth mechanism while maintaining the original physics. Note that very recently we used a specific form of the rescaling scheme to simulate the very long-time dynamics of compact crystals under specialized growth conditions [37,38]. Here, we present a more general version of the scaling scheme, and demonstrate the utility of the scheme in accurately simulating highly ramified Hele–Shaw bubbles over a range of injection fluxes.

By reducing the computation time, this rescaling scheme significantly improves the performance of the boundary integral method originally developed by Hou et al. [9]. In fact, only minor changes to the original algorithm are needed. Using a computer with CPU 2.2 GHz Pentium 4 running Linux (similar to the one used by Fast and Shelley in [39]), we can simulate a high resolution bubble in 6 days that took 50 days for Fast and Shelley [39] to compute. We then continue the simulation significantly longer in time and identify another transition in scaling.

We also investigate the long-time interface morphologies under several injection fluxes $J \propto R(t)^p$ with $p = 1$, 0 and -1, examine the morphologies and measure the bubble complexity in terms of a relation between area $A(t)$ and arclength $L(t)$, i.e. $A(t) \sim L(t)^\gamma$. The relation reflects the underlying physics: when $\gamma < 2$, the destabilizing driving force (flux) dominates the evolution and leads to ramified (e.g. fractal-like) shapes, the smaller the power, the more complex the shape; when $\gamma = 2$, the stabilizing force (surface tension) and destabilizing