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Application of the parabolic spline method (PSM) to a multi-dimensional conservative semi-Lagrangian transport scheme (SLICE) ☆

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Abstract

The recently devised one-dimensional parabolic spline method (PSM) for efficient, conservative, and monotonic remapping is introduced into the semi-Lagrangian inherently-conserving and efficient (SLICE) scheme for transport problems in multi-dimensions. To ensure mass conservation, an integral form of the transport equation is used rather than the differential form of classical semi-Lagrangian schemes. Integrals within the SLICE scheme are computed using multiple sweeps of PSM along flow-dependent cascade directions to avoid the large timestep-dependent splitting errors associated with traditional fixed-direction splitting. Accuracy of the overall scheme, including at large timestep, is demonstrated using two-dimensional test problems in both Cartesian and spherical geometries and compared with that of the piecewise parabolic method (PPM) applied within the same SLICE framework.

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Keywords: Advection; Cascade; Conservation; Monotonicity; PPM; Remapping

1. Introduction

Semi-Lagrangian (SL) schemes [1] are widely used in atmospheric modelling due to their improved stability compared to their Eulerian counterparts, and to the substantial computational savings concomitant with using large timesteps. However, unlike some Eulerian schemes, the lack of mass conservation with SL schemes can be problematic for relatively long-time integrations, such as those for climate studies [2].

The lack of formal mass conservation in SL schemes has been dealt with by either: (i) applying a posteriori corrections, whereby the original global mass is restored by redistributing the deficit/surplus to minimally change the solution [3] (similar approaches have also been used for non-conservative Eulerian schemes [4]);

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or (ii) using inherently conserving schemes, whereby the conservation constraint is an integral part of the scheme, i.e. conservative remapping [5–16]. Such schemes are finite/control-volume methods, in that they are all based on estimating integrals of the conserved quantity over a deformed Lagrangian volume (see Section 2 for details). Recently, Lauritzen [17] has given an analysis of some of these schemes.

Although inherently conservative SL schemes are mathematically well formulated, they tend to be more expensive and difficult to generalise to higher dimensions without a substantial increase in computational cost. Therefore, much of the research in this area has been centered on how to remap a multi-dimensional field in an efficient way to allow the flexibility of using higher-order schemes but without a prohibitive computational overhead. To overcome these conflicting criteria, SLICE [18,19] combines a piecewise cubic method (PCM), which is a higher-order alternative to the popular piecewise parabolic method (PPM [20]), with the cascade (flow-dependent decomposition) approach [21]. Building on the previous development of this scheme, a more efficient variant of SLICE, based on the parabolic spline method (PSM) [22], is presented herein. An advantage of PSM is that of all piecewise parabolic functions that satisfy a given mass (average density) distribution, such as the one used by PPM, PSM is an optimal reconstruction since it possesses the minimum norm (or curvature) and best approximation properties [22]. Furthermore, an operation count shows that PSM is 60% more efficient than PPM, and its monotonic filter damps less than PPM's.

The purpose of the present work is to: (i) outline how the one-dimensional (1D) PSM algorithm (and also the PPM algorithm) can be exploited in multi-dimensions using the SLICE cascade directional decomposition strategy; and (ii) demonstrate that PSM's 1D accuracy advantages over PPM also hold for typical 2D test problems of the literature in both Cartesian and spherical geometries.

The rest of the paper is organised as follows: Section 2 outlines the strategy for incorporating the 1D PSM [22] and PPM [20] remappings into the SLICE cascade framework; in Section 3 results of several illustrative tests in Cartesian and spherical geometry are given; and conclusions are summarised in Section 4.

2. 2D remapping with SLICE

This section briefly outlines how the 1D PSM [22] and PPM [20] remappings can be efficiently incorporated into a general strategy to solve higher dimensional (here 2D) problems by using the SLICE methodology [18,19].

2.1. 2D advective transport

Consider (see e.g. [11]) passive 2D advective transport of a scalar quantity ρ governed, in the absence of sources and sinks, by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{2.1}$$

where ρ is the density (amount of scalar per unit volume) of the transported quantity, and $\mathbf{u}$ and t are the transporting velocity field and time, respectively. Let $\delta\mathscr{V}$ be a material volume (strictly an area here as only two dimensions are considered) that moves with the fluid. Then an equivalent integral form of the conservation equation (2.1) is

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\int_{\delta\mathscr{V}} \rho \, \mathrm{d}\mathscr{V}\right) = 0, \tag{2.2}$$

where $\mathrm{D}/\mathrm{D}t$ is the total derivative following the fluid. Integrating (2.2) from time t_0 to time t_1 gives

$$M_1 = M_0, \tag{2.3}$$

where $M_1 \equiv \int_{\delta\mathscr{V}_1} \rho \, \mathrm{d}\mathscr{V}$, $M_0 \equiv \int_{\delta\mathscr{V}_0} \rho \, \mathrm{d}\mathscr{V}$, and $\delta\mathscr{V}_1$ is the fluid volume at time t_1 that corresponds to the volume $\delta\mathscr{V}_0$ at time t_0. Now let $\delta\mathscr{V}_1$ be a known grid volume, fixed in time, with $\overline{\rho}_1$ being the associated value of the scalar averaged over this volume, so that $M_1 = \overline{\rho}_1 \delta\mathscr{V}_1$. In this context $\delta\mathscr{V}_1$ is then an Eulerian control volume (ECV) whereas $\delta\mathscr{V}_0$ is its corresponding Lagrangian control volume (LCV). Then the problem simplifies to computing the discrete integral M_0, which is a remapping of a given average field $\overline{\rho}$ at time t_0 on regular ECV's to irregular LCV's.