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Fast spin ±2 spherical harmonics transforms and application in cosmology

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Abstract

A fast and exact algorithm is developed for the spin ±2 spherical harmonics transforms on equi-angular pixelizations on the sphere. It is based on the Driscoll and Healy fast scalar spherical harmonics transform. The theoretical exactness of the transform relies on a sampling theorem. The associated asymptotic complexity is of order $\mathcal{O}(L^2\log_2^2L)$, where $2L$ stands for the square-root of the number of sampling points on the sphere, also setting a band limit L for the spin ±2 functions considered. The algorithm is presented as an alternative to existing fast algorithms with an asymptotic complexity of order $\mathcal{O}(L^3)$ on other pixelizations. We also illustrate these generic developments through their application in cosmology, for the analysis of the cosmic microwave background (CMB) polarization data.

Keywords: Computational methods; Data analysis; Cosmology; Cosmic microwave background

1. Introduction

In the last few years, the analysis of the temperature anisotropies of the cosmic microwave background (CMB), together with other cosmological observations, has allowed the definition of a precise concordance cosmological model. These observations culminated with the release of the three-year data of the Wilkinson Microwave Anisotropy Probe (WMAP) satellite experiment. The cosmological parameters are now determined with an unprecedented precision of the order of several percent [1–4]. In the concordance model, the CMB originates from quantum energy fluctuations defined in a primordial era of inflation. These tiny fluctuations are Gaussian in first approximation. The cosmological principle of homogeneity and isotropy of the universe is also assumed. The observed radiation is therefore understood as a unique realization of a Gaussian and stationary (*i.e.* homogeneous and isotropic) random process on the sphere, which may be completely characterized from its two-point correlation functions, or the corresponding angular power spectra.

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The present concordance values of the cosmological parameters are obtained through a best fit of the theoretical temperature angular power spectrum of the CMB with the experimental data. Beyond temperature anisotropies, *i.e.* intensity anisotropies, a polarization of the CMB is also present which constitutes a complementary source of information for cosmology. This polarization is produced through Thomson scattering at the epoch of recombination. The degree of polarization of the CMB is expected to be of the order of 10 percent of the temperature anisotropies at small scales, and lower at large scales. As Thomson scattering only produces linearly polarized light, the CMB radiation is completely described by its temperature T, and its linear polarization Stokes parameters Q and U [5–10]. First polarization measurements were recently obtained, notably by the WMAP experiment [11]. Future CMB experiments such as the Planck surveyor satellite experiment will allow a deeper probe of the temperature and polarization spectra, thanks to improved sensitivity and resolution on the whole sky.

From the mathematical point of view, the observable temperature T is a scalar function on the sphere, *i.e.* invariant under local rotations in the plane tangent to the sphere at each point. The associated invariant TT angular power spectrum results from the decomposition of the temperature in scalar spherical harmonics. But the observable polarization Stokes parameters Q and U transform as the components of a transverse, symmetric, and traceless rank 2 tensor under local rotations. However, scalar electric E and magnetic B polarization components may equivalently be defined from the parameters Q and U. The associated invariant EE and BB polarization angular power spectra, and the cross-correlation TE spectrum result from the decomposition of the combinations $Q \pm \mathrm{i}U$ in spin ± 2 spherical harmonics on the sphere [6]. From the numerical point of view, the asymptotic complexity associated with a naive quadrature based on the definition of the scalar and spin ± 2 spherical harmonics transforms is of order $\mathcal{O}(L^4)$, where L roughly identifies the square-root of the number of sampling points on the sphere. Corresponding computation times for the analysis of megapixels all-sky maps such as those of the ongoing WMAP or the forthcoming Planck experiments are of the order of days. Fast and precise computation methods for the scalar and spin ± 2 spherical harmonics transforms of functions on the sphere are therefore needed.

Beyond cosmology, an algorithm for the spin ± 2 spherical harmonics transforms will find application in the spectral analysis of arbitrary spin ± 2 signals on the sphere, components of transverse, symmetric, and traceless rank 2 tensor fields under local rotations.

In the present work we develop a fast algorithm for the spin ± 2 spherical harmonics transforms of band-limited functions on the sphere. It is based on an existing fast algorithm for the scalar spherical harmonics transform. It is defined on $2L \times 2L$ equi-angular pixelizations in spherical coordinates (θ, φ) on the sphere. The algorithm is theoretically exact thanks to the existence of a sampling theorem. The associated asymptotic complexity is of order $\mathcal{O}(L^2 \log_2^2 L)$. Corresponding computation times for megapixels maps are reduced to seconds. The algorithm is presented as an alternative to existing fast algorithms with an asymptotic complexity of order $\mathcal{O}(L^3)$ on other pixelizations which are widely used in the context of astrophysics and cosmology.

In Section 2, we recall the notion of spin n functions on the sphere. In Section 3, we define and implement a fast and exact algorithm with complexity $\mathcal{O}(L^2 \log_2^2 L)$ for the spin ± 2 spherical harmonics transforms on equi-angular pixelizations. In Section 4, we illustrate the interest of our algorithm in the context of the analysis of CMB polarization data. We finally briefly conclude in Section 5.

2. Spin *n* functions on the sphere

In this section, we discuss standard harmonic analysis on the sphere and on the rotation group $SO(3)$. We also discuss the notion of spin n functions on the sphere and their decomposition in a basis of spin-weighted spherical harmonics of spin n.

2.1. Standard harmonic analysis

Let the function $G(\omega)$ be a square-integrable function in $L^2(S^2, \mathrm{d}\Omega)$ on the unit sphere S^2. The spherical coordinates of a point on the unit sphere, defined in the right-handed Cartesian coordinate system $(o, o\hat{x}, o\hat{y}, o\hat{z})$ centered on the sphere, read as $\omega = (\theta, \varphi)$. The angle $\theta \in [0, \pi]$ is the polar angle, or co-latitude. The angle $\varphi \in [0, 2\pi]$ is the azimuthal angle, or longitude. The invariant measure on the sphere reads