Journal of Computational Physics 229 (2010) 6104–6134

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

High-order finite-volume methods for the shallow-water equations on the sphere

Paul A. Ullrich [a,*], Christiane Jablonowski [a], Bram van Leer [b]

[a] Department of Atmospheric, Oceanic and Space Sciences, Space Research Building, University of Michigan, 2455 Hayward St., Ann Arbor, MI 48109, United States
[b] Department of Aerospace Engineering, François-Xavier Bagnoud Building, University of Michigan, 1320 Beal Avenue, Ann Arbor, MI 48109, United States

ARTICLE INFO

Article history:
Received 6 October 2009
Received in revised form 16 April 2010
Accepted 27 April 2010
Available online 4 May 2010

Keywords:
High-order
Finite-volume methods
Cubed-sphere
Shallow-water equations
Riemann solvers
Well-balanced schemes

ABSTRACT

This paper presents a third-order and fourth-order finite-volume method for solving the shallow-water equations on a non-orthogonal equiangular cubed-sphere grid. Such a grid is built upon an inflated cube placed inside a sphere and provides an almost uniform grid point distribution. The numerical schemes are based on a high-order variant of the Monotone Upstream-centered Schemes for Conservation Laws (MUSCL) pioneered by van Leer. In each cell the reconstructed left and right states are either obtained via a dimension-split piecewise-parabolic method or a piecewise-cubic reconstruction. The reconstructed states then serve as input to an approximate Riemann solver that determines the numerical fluxes at two Gaussian quadrature points along the cell boundary. The use of multiple quadrature points renders the resulting flux high-order. Three types of approximate Riemann solvers are compared, including the widely used solver of Rusanov, the solver of Roe and the new AUSM$^+$-up solver of Liou that has been designed for low-Mach number flows. Spatial discretizations are paired with either a third-order or fourth-order total-variation-diminishing Runge–Kutta timestepping scheme to match the order of the spatial discretization. The numerical schemes are evaluated with several standard shallow-water test cases that emphasize accuracy and conservation properties. These tests show that the AUSM$^+$-up flux provides the best overall accuracy, followed closely by the Roe solver. The Rusanov flux, with its simplicity, provides significantly larger errors by comparison. A brief discussion on extending the method to arbitrary order-of-accuracy is included.

1. Introduction

Atmospheric models are difficult to engineer, largely due to two factors. Firstly, the flow occurs over the surface of a sphere, rather than in much simpler planar Cartesian geometry and secondly, there are vast scale differences between the large-scale horizontal flow, with length scales that extend to thousands of kilometers, and vertical motions with length scales of about 1–10 km. In addition, the dominant motions in the atmosphere are an example of a low-Mach number regime that is mostly characterized by Mach numbers around $M < 0.4$. Therefore, care must be taken when applying numerical methods from other research fields. In particular, in atmospheric flows high-speed motions are only present in fast atmospheric gravity waves or sound waves. The latter are a solution to the 3D nonhydrostatic equation set, but play a negligible role from a physical viewpoint. Nevertheless, an adequate numerical scheme for atmospheric flows must guarantee stability for fast waves and treat the slow, physically important, motions with high accuracy.

* Corresponding author. Tel.: +1 734 936 5503; fax: +1 734 763 0437.
E-mail addresses: paullric@umich.edu (P.A. Ullrich), cjablono@umich.edu (C. Jablonowski), bram@umich.edu (B. van Leer).

0021-9991/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcp.2010.04.044

A common test bed for atmospheric model development is based on the shallow-water equation set that mimics atmospheric flow in a single layer. A shallow-water model thereby tests the horizontal and temporal discretizations and provides guidance for the numerical schemes suitable for flows with low-Mach numbers. Note that the shallow-water equations do not support sound waves but do capture the fast gravity wave propagation.

There are many numerical schemes that have been tested in shallow-water models on the sphere, all of which have both pros and cons. The spectral transform method discussed in Jakob-Chien et al. [17] achieves high accuracy but tends to exhibit non-physical numerical oscillations near sharp gradients – so-called Gibb's ringing. Spectral transform methods also demand a high computational expense at high resolution that is associated with the computational cost of the Legendre transforms. Finite-difference approaches include those of Heikes and Randall [16] and Ronchi et al. [32]. Hybrid finite-volume methods incorporate both a finite-volume treatment of conservative variables and a finite-difference treatment of momentum and include the models of Lin and Rood [22] and Chen and Xiao [8]. Finite-element type models, including spectral-element (SE) and discontinuous-Galerkin (DG) models have been presented by Taylor et al. [37], Côté and Staniforth [10], Thomas and Loft [39], Giraldo et al. [13] and Nair et al. [26].

The aforementioned models represent a wide variety of computational grids on the sphere such as the latitude–longitude mesh, icosahedral and hexagonal grids, and cubed-spheres meshes. The latter three have become popular over the last decade as they provide an almost regular grid point coverage on the sphere. The uniform distribution of elements avoids the convergence of the meridians that is characteristic for latitude–longitude grids, and thereby alleviates the use of polar filters and other numerical damping techniques. The cubed-sphere grid has also been proven to scale efficiently on massively parallel computing platforms as shown by Taylor et al. [38] and Putman and Lin [30]. These two models are therefore under consideration for operational climate and weather applications at atmospheric modeling centers in the US.

This paper introduces a set of third- and fourth-order-accurate fully-conservative finite-volume methods on cubed-sphere grids and assesses the impact of the high-order accuracy. These finite-volume methods are built upon the reconstruction techniques adopted by the Monotone Upstream-centered Schemes for Conservation Laws (MUSCL) pioneered by Van Leer [44]. Previously, second-order finite-volume methods of this type have been studied for geostropic flows on the sphere by Rossmanith [33], which is based on the flux-difference-splitting technique of LeVeque [20] on a curved manifold.

Fully-conservative finite-volume methods share local conservation properties with spectral-element and discontinuous-Galerkin discretizations, but are potentially more computationally efficient due to their relatively weak Courant–Friedrichs–Lewy (CFL) constraints. Explicit timestepping techniques, when used in combination with these methods, suffer from severe CFL timestep restrictions related to the clustering of nodal points near element edges (which worsens at high-order). On the other hand, finite-volume methods possess a large computational stencil at high-order and so are also potentially difficult to parallelize as effectively as these more compact methods. This difficulty arises primarily in the algorithmic complexity associated with determining which information needs to be communicated between processors. Although DG and SE methods only require information to be communicated between elements and their immediate neighbors, the number of prognotistic quantities associated with each element is significantly larger for these schemes. Whereas for each state variable finite-volume methods store only one value per element, DG and SE methods can, at fourth-order-accuracy, can have up to 10 values per element.

The use of neighboring elements by high-order FV schemes also means that element values must be remapped across coordinate discontinuities, such as those that appear on the cubed-sphere grid. This requirement results in the need for wider ghost regions near coordinate discontinuities on parallel systems in order to accommodate remapping. Schemes with local degrees of freedom, on the other hand, including DG and SE methods, may be more attractive in this regard since remapping is not required, and hence work can be distributed more evenly on parallel architectures.

Although we do not present a technique for constructing a monotone or non-oscillatory scheme in this paper, significant research has been done on this topic for applications in other research areas. For instance, (Weighted) Essentially Non-Oscillatory ((W) ENO)-type reconstructions (e.g. [2,27]), slope limiters (e.g. [43,25]) or flux-corrected transport methods [49] can all be applied to this class of finite-volume methods presented herein. Monotone DG methods, on the other hand, are an active research area.

The method we present involves the use of approximate Riemann solvers to calculate edge fluxes. Most widely used approximate Riemann solvers (such as the solver of Roe [31]) are designed to model flow in the transsonic or supersonic regime rather than in the relatively slow flow regime that is typical for the atmosphere. However, recent advances in the design of approximate Riemann solvers have led to an extension of the Advection Upstream Splitting Method (AUSM, [24]) to low Mach numbers [23]. The use of this new numerical flux formulation, known as AUSM$^+$-up, has so-far been largely limited to the aerospace community. Hence, a test of this new approximate Riemann solver will gauge its applicability for atmospheric models. We also compare the Roe and AUSM$^+$-up schemes to the widely-used and simpler Rusanov solution [19,34,42].

The performance of all schemes will be analyzed via selected standard test cases from the suite of Williamson et al. [46]. Among them are the advection of a cosine bell, steady-state geostrophic flow, steady-state geostrophic flow with compact support, flow over an isolated mountain and the Rossby–Haurwitz wave. In addition, we assess the barotropic instability problem of Galewsky et al. [12] that exhibits sharp vorticity gradients.

The paper is organized as follows: In Section 2 we introduce the cubed-sphere grid with an equiangular projection, which is the underlying grid for all simulations. Section 3 discusses the shallow-water equations in cubed-sphere geometry. The high-order finite-volume framework is described in Section 4. Special attention is paid to a careful discretization of the