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Limiters for high-order discontinuous Galerkin methods

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Abstract

We describe a limiter for the discontinuous Galerkin method that retains as high an order as possible, and does not automatically reduce to first order. The limiter is a generalization of the limiter introduced in [R. Biswas, K. Devine, J.E. Flaherty, Parallel adaptive finite element methods for conservation laws, Applied Numerical Mathematics 14 (1994) 255–284]. We present the one-dimensional case and extend it to two-dimensional problems on tensor-product meshes. Computational results for examples with both smooth and discontinuous solutions are shown.

Keywords: Limiters; High-resolution schemes; Discontinuous Galerkin methods; Euler equations

1. Introduction

Discontinuous Galerkin (DG) methods are becoming popular due to the ease of increasing the order of approximation while keeping the stencil local. They combine the ease of finite element approximations in handling complex geometry and adaptation with the shock-capturing abilities of finite volume schemes. One aspect of these methods, however, that is not yet satisfactory is limiting. When a DG solution is limited, most methods reduce the solution to first-order accuracy, and much of the advantage of high-order methods is lost.

Some form of nonlinear limiting seems necessary in high-order computations of discontinuous flows [9,22,10]. We propose a limiter for use with the DG schemes for hyperbolic conservation laws that can limit gradually, systematically reducing the order of accuracy depending on the behavior of the higher-order solution derivatives. It does not automatically reduce to first order. We then extend it to two-dimensional problems on tensor-product meshes. The limiter is a generalization of the *moment* limiter proposed by Biswas et al. [3].

The moment limiter itself is a generalization of the second-order accurate minmod limiter of van Leer [22] to higher orders of approximation. The minmod limiter reduces the slope in a cell if the solution in that cell exceeds the range of solution averages on neighboring cells. The moment limiter works in a similar way: it

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limits the derivative of order i in a given cell using the derivatives of order $i-1$ in neighboring cells. As with van Leer type limiters, the strength of the moment limiter can be varied.

With DG methods, high-order limiting on general meshes remains an open question both from the theoretical and practical points of view. DG methods achieve formal order of accuracy $p+1$ by representing the solution as a polynomial of degree p in each computational cell. In the absence of high-order limiters, alternative techniques have been developed to control oscillations in approximations of order $p>1$. There are a number of discontinuity detection strategies where discontinuities are first detected and then a limiter (usually minmod) is applied only on the elements that are believed to contain a discontinuity. For an overview and comparison of such strategies, see [18]. Qiu et al. [19] proposed using a high-order WENO reconstruction instead of the minmod limiter in conjunction with a discontinuity detection strategy. Jaffre et al. [14] introduced the idea of adding artificial viscosity as a stabilization tool. The amount of viscosity is based on the size of the residual. An implementation of this approach can be found in [12]. More recently an artificial viscosity term based on h and p was used in [17], also in conjunction with a discontinuity detection strategy and sub-cell resolution. Modal filtering has been successfully applied to a nodal based DG, see for example [8]. Finally, Hoteit et al. [13] developed a limiter applicable to piecewise quadratic solutions on rectangular meshes. The limiter is vertex based: it requires the degree of freedom associated with a vertex to lie between the cell averages of all elements containing the vertex. A minimization problem needs to be solved on each cell. In contrast to these approaches, our limiter is closer in spirit to those used in finite volume schemes.

Our limiter is applied progressively, limiting first the high-order terms as needed (e.g. as the solution starts to steepen). The process continues until either a coefficient is found that does not need to be limited or all terms are limited. This has two beneficial effects. First, it achieves the highest possible accuracy when some limiting is necessary. Second, gradual introduction of the limiter seems to avoid artificial limiter-induced steepening – turning sine waves into square waves – that happens with some limiters.

One reason for the absence of high-order limiters might be the lack of analytical tools. The total variation diminishing (TVD) theory of Harten [10] has been very powerful in constructing second-order limiters in one space dimension. Harten looks at the total variation of the solution means

$$TV = \sum_j |\overline{U}_{j+1} - \overline{U}_j|, \tag{1}$$

which should be non-increasing with time. However, such schemes reduce to first-order near smooth extrema [11]. This leads to the conclusion that all TVD schemes are at most second-order accurate. However, there are a couple of interesting constructions for piecewise parabolic solutions of scalar problems in one dimension [20,16] that are TVD in a different sense. They measure the total variation of the entire function, consisting of the variation of the solution within mesh cells, and including the jumps between cells. It is interesting to note that with the minmod limiter, linear DG solutions in one dimension are TVD in means and are not TVD in this more general sense (see Example 4.1.2).

The limiter that we propose is not total variation diminishing in either sense. In our experiments, the solutions are total variation bounded, but we are unable to prove this analytically. The adaptive character of the limiter (we stop if a coefficient is found that does not need to be limited) makes it difficult to analyze. We should note that some commonly used schemes such as ENO/WENO [11,15] and PPM [6] are not provably TVD, but also seem to be nonlinearly stable.

The outline of the remainder of the paper is as follows. In Section 2 we present the moment limiter in the one-dimensional case, for both scalar equations and systems of equations. Section 3 extends the limiter to two space dimensions. Computational results on a variety of test cases in both one and two dimensions are presented in Section 4 with conclusions following in Section 5. We largely omit a description of local Runge–Kutta DG schemes. Classical papers of Cockburn and Shu are a good reference [5,4]; details of the specific implementation used by the author can be found in [7].

2. One-dimensional limiting

In this section, we present the moment limiter, which is a generalization of the minmod limiter. We will argue that the ith derivative of the numerical solution should not exceed forward and backward differences