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Short Note

A comment on the computation of non-conservative products

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ABSTRACT

We are interested in the solution of non-conservative hyperbolic systems, and consider in particular the so-called path-conservative schemes (see e.g. [2,3]) which rely on the theoretical work in [1]. The example of the standard Euler equations for a perfect gas is used to illuminate some computational issues and shortcomings of this approach.

1. Introduction

Non-conservative hyperbolic systems arise in a wide range of applications, which makes their theoretical study and numerical approximation a very important topic. We are interested in the numerical solution of non-conservative hyperbolic systems

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0, \tag{1}$$

subject to initial conditions. Here $U \in \Omega \subset \mathbb{R}^p$.

The challenge here is twofold: first, to generalize the notion of weak solutions to the case where the underlying hyperbolic system is not in conservation form, by giving an acceptable definition of shock waves. Second, once the theoretical framework has been established, to compute solutions to those systems. The difficulty lies in the fact that while for conservative systems, shock relations depend on the solution states to the immediate left/right of the shock, in the non-conservative case they depend not only on those states but also on the path that connects them

$$\sigma(U_R - U_L) = \int_{U_L}^{U_R} A(U)\, dU,$$

here σ denotes the shock speed, and $U_{L,R}$ the left/right states. In the conservative case, $A(U)$ is the Jacobian of a flux function $F(U)$ and the above relation recovers the standard Rankine–Hugoniot relations.

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There has been many contributions on the topic during recent years. From the theoretical point of view, most notably the work of Dal Maso et al. [1] where the notion of path is introduced to define generalized shock relations: given a family of paths, i.e. C^1 function from $[0,1]\times\Omega^2\mapsto\Omega$

$$(x,u,v)\mapsto\Phi(z;u,v),$$

such that $\Phi(0,u,v)=u$ and $\Phi(1;u,v)=v$. and given two states U_L and U_R, the two states are said to define a shock moving at speed σ if

$$\sigma(U_L-U_R)=\int_0^1 A(\Phi(s,;U_L,U_R))\frac{\partial\Phi(s;U_L,U_R)}{\partial s}ds. \tag{2}$$

With this relation, a mathematical theory for weak solutions for hyperbolic systems in non-conservation form is developed, and the solution to the Riemann problem may be constructed [1].

Once the Rankine–Hugoniot relations have been encoded by (2), a numerical approximation of (1), called path-conservative, have been proposed by Parés and collaborators (see for example [2,3]). Given $\Delta x>0$ and the mesh $\{x_j\}_{j\in\mathbb{Z}}$ with $x_j=j\Delta x$, the path-conservative schemes are defined as schemes of the form

$$U_j^{n+1}=U_j^n-\frac{\Delta t}{\Delta x}\left(D^-_{j+1/2}(U_j^n,U_{j+1}^n)+D^+_{j-1/2}(U_{j-1}^n,U_j^n)\right) \tag{3}$$

where the residuals $D^\pm_{j+1/2}(U_j^n,U_{j+1}^n)$ satisfy the conservation relation

$$D^-_{j+1/2}(U_j^n,U_{j+1}^n)+D^+_{j+1/2}(U_j^n,U_{j+1}^n)=\int_0^1 A\big(\Phi(s,;U_j,U_{j+1})\big)\frac{\partial\Phi(s;U_L,U_R)}{\partial s}ds. \tag{4}$$

In [2], the theoretical framework for path-conservative schemes was presented, and a Lax-Wendroff theorem was conjectured: if the numerical solution $(U_j^n)_{j,n}$ obtained by a path-conservative scheme converges, its limit is the weak solution in the sense defined by the theory of [1]. A more recent paper [3] found that in fact those schemes generate convergence error source-term which is supported on shock trajectories and that the error measure is usually 'small'. The paper presented a thorough numerical investigation to evaluate the range of validity of certain schemes.

Several questions seem legitimate: (i) how does one go about choosing a path; (ii) what influence does the choice of path and discretization scheme have on the computed solution; (iii) once a path is specified and a consistent path-conservative scheme designed, does the numerical solution converge to the assumed path; and (iv) in cases where the correct jump conditions are known unambiguously, can a path-conservative scheme be designed so that it converges to the correct solution?

The answers to these questions are in general quite difficult, so we proceed by considering the following illuminating example.

2. A simple example

Consider the Euler equations of fluid dynamics in Lagrangian coordinates

$$\begin{aligned} &v_t-u_m=0,\\ &u_t+p_m=0,\\ &e_t+(pu)_m=0. \end{aligned} \tag{5}$$

Here, v is the specific volume, u the velocity, e the specific total energy and p is the pressure. We also use $e=\varepsilon+u^2/2$, where ε denotes the specific internal energy. The equation of state for perfect gas is given by

$$\varepsilon=(\gamma-1)pv$$

with γ the specific heat ratio, taken to be 1.4 in the numerical experiments.

We also write (5) in a non-conservative manner, in terms of (v,u,ε)

$$\begin{aligned} &v_t-u_m=0,\\ &u_t+p_m=0,\\ &\varepsilon_t+pu_m=0. \end{aligned} \tag{6}$$

We observe that (6) is 'minimally' non-conservative in that it only has *one* non-conservative product, and otherwise has a conservative sub-system in the first two equations. We consider both systems (5) and (6) and try to shed light on the questions raised in the previous section. Of course, in this case, the correct shock relations are given by the conservative system (5). In designing numerical approximations to solve (6), the ultimate task is to compute solutions of (6) which recover the shock relations of (5).