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Dispersion and dissipation properties of the 1D spectral volume method and application to a *p*-multigrid algorithm

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Abstract

In this article, the wave propagation properties of the 1D spectral volume method are studied through analysis of the Fourier footprint of the schemes. A *p*-multigrid algorithm for the spectral volume method is implemented. Restriction and prolongation operators are discussed and the efficiency of the smoothing operators is analyzed. The results are verified for simple 1D advection problems and for a quasi-1D Euler flow. It is shown that a significant decrease in computational effort is possible with the *p*-multigrid algorithm.

Keywords: Spectral volume method; Wave propagation properties; *p*-Multigrid

1. Introduction

The spectral volume method was developed in a series of papers by Wang et al. [1–4], for the solution of general two-dimensional hyperbolic systems like the Euler equations and on unstructured triangular meshes. Recently, Liu et al. [5] extended the method to general three-dimensional hyperbolic systems on unstructured tetrahedral meshes and Sun et al. [6] implemented the method for the Navier–Stokes equations. The spectral volume method can be interpreted as a finite volume method, where within each spectral volume (SV) cell a miniature structured mesh of control volumes (CVs) is formed. In this way a unique stencil for the flux is defined for each face, eliminating the need for searching operations, which are needed in traditional high-order (>2) finite volume methods on unstructured grids. Alternatively, the method can be interpreted as a Petrov–Galerkin method. The weighting functions then correspond to Heaviside-like functions, which are equal to one within one CV and zero everywhere else in the SV. The trial functions are chosen such that they have an average value equal to one in one CV and average values of zero in the other CVs. The method enjoys many of the advantageous properties of the discontinuous Galerkin method. It is capable of achieving arbitrarily high orders of accuracy on unstructured grids. It also has a compact stencil, since a SV only communicates

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with its immediate neighbours, through the use of Riemann fluxes, which makes the method suited for parallel computations. The representation of the solution within a SV, by averages over the CVs, allows for a limiting procedure on a control volume level. This gives the SV method a higher resolution for shocks than the discontinuous Galerkin method.

In previous work [2,3,5], the Lebesgue constant, which is basically a measure for the quality of the polynomial interpolation, was used to develop SV partitions. In this paper, the dispersive and dissipative errors of the schemes, which are very important for applications such as Computational Aeroacoustics (CAA) and Large Eddy Simulation (LES), will be used to analyze the performance of the SV partitions. Such analysis is widely used for finite difference and finite volume methods, and Hu et al. [9] performed it for the discontinuous Galerkin method.

With high-order methods, it is possible to achieve low error levels more efficiently than with traditional first-order and second-order accurate schemes. However, efficient solution methods are necessary to fully fulfill this potential. Multigrid algorithms are by far the best candidate for this purpose. Traditional *h*-multigrid is nowadays a standard tool for CFD-applications. In the last few years, *p*-multigrid algorithms for discontinuous Galerkin methods have been investigated and successfully implemented by Helenbrook et al. [12], Bassi and Rebay [13] and Fidkowski et al. [14,15]. Important speed-ups in convergence were observed. In this paper, a *p*-multigrid implementation for the SV method and adapted Runge–Kutta solvers for optimal smoothing properties are presented.

The paper is organized as follows. Section 2 gives a brief description of the SV method. In Section 3, the dispersive and dissipative properties of the SV method are analyzed through the Fourier footprint. The *p*-multigrid algorithm is then described in Section 4. In Section 5, the Fourier footprints are used to investigate the efficiency of the smoothing operators for the multigrid algorithm. Numerical results are presented in Section 6. Finally, conclusions are drawn in Section 7.

2. The spectral volume method

The spectral volume method can be applied to hyperbolic conservation laws (1)

$$\frac{\partial U}{\partial t} + \vec{\nabla} \cdot \vec{F}(U) = S(U) \tag{1}$$

The computational domain V is divided in N_{SV} spectral volumes (SV) V_i with volume $|V_i|$. Each of these SVs V_i is further subdivided into control volumes (CV) $V_{i,j}$. Integrating (1) over such a CV and applying the Gauss theorem gives

$$\frac{\partial \overline{U}_{i,j}}{\partial t}|V_{i,j}| = -\int_{\partial V_{i,j}} \vec{F} \cdot \mathrm{d}\vec{s} + \int_{V_{i,j}} S \,\mathrm{d}V = R_{i,j} \tag{2}$$

where $|V_{i,j}|$ is the volume of $V_{i,j}$, $R_{i,j}$ is the residual corresponding to $V_{i,j}$ and $\overline{U}_{i,j}$ is the CV average defined by

$$\overline{U}_{i,j} \equiv \frac{1}{|V_{i,j}|} \int_{V_{i,j}} U \,\mathrm{d}V \tag{3}$$

On a spectral element V_i, a polynomial approximation of the solution is defined

$$U_{V_i} \approx u_{V_i} \equiv \sum_{j=1}^{N_i(p,d)} \overline{U}_{i,j} L_{i,j} \tag{4}$$

$N_i(p,d)$ is the number of CVs in the SV V_i, depending on the desired degree of the polynomial approximation p and the number of spatial dimensions d. The polynomials $L_{i,j}$ associated to the CVs $V_{i,j}$ are defined by

$$\frac{1}{|V_{i,j}|} \int_{V_{i,j}} L_{i,m} \,\mathrm{d}V = \delta_{jm} \tag{5}$$

where δ_{jm} is the Kronecker delta function. Eq. (5) ensures the following property of the polynomial approximation