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Matched interface and boundary (MIB) method for elliptic problems with sharp-edged interfaces

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Abstract

Elliptic problems with sharp-edged interfaces, thin-layered interfaces and interfaces that intersect with geometric boundary, are notoriously challenging to existing numerical methods, particularly when the solution is highly oscillatory. This work generalizes the matched interface and boundary (MIB) method previously designed for solving elliptic problems with curved interfaces to the aforementioned problems. We classify these problems into five distinct topological relations involving the interfaces and the Cartesian mesh lines. Flexible strategies are developed to systematically extends the computational domains near the interface so that the standard central finite difference scheme can be applied without the loss of accuracy. Fictitious values on the extended domains are determined by enforcing the physical jump conditions on the interface according to the local topology of the irregular point. The concepts of primary and secondary fictitious values are introduced to deal with sharp-edged interfaces. For corner singularity or tip singularity, an appropriate polynomial is multiplied to the solution to remove the singularity. Extensive numerical experiments confirm the designed second order convergence of the proposed method.

Keywords: Immersed boundary method; Immersed interface method; Ghost fluid method; Elliptic equations; Sharp-edged interfaces

1. Introduction

Since the pioneer work of Peskin [50] in 1977, much attention has been paid to the numerical solution of elliptic equations with discontinuous coefficients and singular sources on regular Cartesian grids [7,8,11,15,18,20,30–32,55,58]. Simple Cartesian grids are preferred in these studies since the complicated procedure of generating unstructured grid could be bypassed, and well developed fast algebraic solvers could be utilized. The importance of elliptic interface problems has been well recognized in a variety of disciplines, such as fluid dynamics [16,19,29,47], electromagnetics [23,24] and material science [26]. However, to construct

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highly efficient methods for these problems is a difficult task due to the low global regularity of the solution. Traditional numerical methods that are constructed with the assumption of smooth solutions cannot perform at designed accuracy, and might even diverge. For this class of problems, apart from Peskin's immersed boundary method (IBM) [21,33,50–52], a number of other elegant methods have been proposed. Among them, the immersed interface method (IIM), proposed by LeVeque and Li [35] is a second order sharp interface scheme. The IIM has been made robust and efficient over the past decade [1,14,36,37,54]. The ghost fluid method (GFM) [17] was proposed as a relatively simple and easy to use approach. For irregular interfaces, it is nature to construct a solution in the finite element method formulation [2,9,38], in particular, using the discontinuous Galerkin technique [22]. A relevant, while quite distinct approach is the integral equation method for complex geometry [44,45]. Aforementioned methods have found much success in scientific and engineering applications [6–8,15,18,20,25–28,30,32,34,39,41,40,42,53,54,57–59]. A possible further direction in the field could be the development of higher order interface methods [20,60,61] which are particularly desirable for problems involving both material interfaces and high frequency oscillations, such as the interaction of turbulence and shock, and high frequency wave propagation in inhomogeneous media [5].

One of the most challenging problems in the field is the solution of elliptic equations with sharp-edged coefficients, i.e., non-smooth interfaces. Numerical solutions to this class of problems have widespread applications in science and engineering, such as electromagnetic wave scattering and propagation [12,48,49], wave-guides analysis [46], plasma–surface interaction [43], friction modeling [56] and turbulent-flow [4]. To the best of our knowledge, none of the aforementioned methods proposed for elliptic interface problems have been directly applied to the treatment of sharp-edged interfaces. Essentially, as the gradient near the tips of sharp-edged interface is not well defined, some earlier interface methods might not work. Most existing results on this class of problems are obtained by using finite element methods [46,49]. However, finite element methods might exhibit a reduced convergence rate when used for the analysis of geometries containing sharp edges [25,49]. Consequently, dramatic local mesh refinement is required in the vicinity of sharp edges [13], and leads to severe increase in computational time and memory requirement. In particular, local mesh refinement does not work if the solution is highly oscillatory due to the so-called pollution effect [3], which is a common situation in dealing with electromagnetic wave scattering and propagation. Hou and Liu proposed a finite element formulation [25] for solving elliptic equations with sharp-edged interfaces. Remarkably, these authors have achieved about 0.8th order convergence with non-body-fitting grids.

The objective of the present work is to extend the matched interface and boundary (MIB) method previously designed for solving elliptic problems with curved interfaces to problems with sharp-edged interfaces, thin-layered interfaces and interfaces that intersect with the geometric boundary. The MIB was proposed by Zhao and Wei [60] as a systematic higher-order method for electromagnetic wave propagation and scattering in dielectric media. Recently, it has been generalized for solving elliptic equations with curved interfaces by Zhou et al. [61]. The MIB approach makes use of fictitious domains so that the standard high order central finite difference (FD) method can be applied across the interface without the loss of accuracy. The fictitious values on fictitious domains are determined from enforcing the interface jump conditions at the exact position of the interface. One feature of the MIB is that it disassociates between the discretization of the elliptic equation and the enforcement of interface jump conditions. Another feature is to make repeated use of the lowest order jump conditions to determine the fictitious values on extended domains. Since only lowest order interface jump conditions are repeatedly used in the MIB method, arbitrarily high order convergence can be achieved in principle. For straight interfaces, MIB schemes of up to 16th order have been constructed [60,61]. For lightly curved interfaces, up to 6th order schemes have been demonstrated [61]. Most recently, we have proposed an interpolation formulation of the MIB method without the explicit use of fictitious values [62]. We have shown that our interpolation formulation is equivalent to our earlier fictitious domain formulation. Fourth order convergence is obtained for arbitrarily curved interfaces. In the present work, we further generalize the MIB method to allow the presence of sharp-edged interfaces, thin-layered interfaces and interfaces that intersect with the domain boundary. For these problems, flexible strategies that have not been ever considered before are required. We introduce the concepts of primary and secondary fictitious values to overcome the difficulty of sharp-edged interfaces. The essence is to replace unavailable auxiliary points by secondary fictitious points to resolve primary fictitious values when there are geometric difficulties. The topological relations between the interfaces and the Cartesian mesh lines are classified into five distinct types. For each