Available online at www.sciencedirect.com

ScienceDirect

Journal of Computational Physics 224 (2007) 1021–1048

JOURNAL OF COMPUTATIONAL PHYSICS

www.elsevier.com/locate/jcp

A new adaptive local mesh refinement algorithm and its application on fourth order thin film flow problem

Pengtao Sun [b,*], Robert D. Russell [a], Jinchao Xu [b]

[a] *Department of Mathematics, Simon Fraser University, USA*
[b] *Department of Mathematics, Pennsylvania State University, USA*

Received 23 February 2006; received in revised form 26 September 2006; accepted 4 November 2006
Available online 20 December 2006

Abstract

A new adaptive local mesh refinement method is presented for thin film flow problems containing moving contact lines. Based on adaptation on an optimal interpolation error estimate in the L^p norm ($1 < p \leqslant \infty$) [L. Chen, P. Sun, J. Xu, Multilevel homotopic adaptive finite element methods for convection dominated problems, in: Domain Decomposition Methods in Science and Engineering, Lecture Notes in Computational Science and Engineering 40 (2004) 459–468], we obtain the optimal anisotropic adaptive meshes in terms of the Hessian matrix of the numerical solution. Such an anisotropic mesh is optimal for anisotropic solutions like the solution of thin film equations on moving contact lines. Thin film flow is described by an important type of nonlinear degenerate fourth order parabolic PDE. In this paper, we address the algorithms and implementation of the new adaptive finite element method for solving such fourth order thin film equations. By means of the resulting algorithm, we are able to capture and resolve the moving contact lines very precisely and efficiently without using any regularization method, even for the extreme degenerate cases, but with fewer grid points and degrees of freedom in contrast to methods on a fixed mesh. As well, we compare the method theoretically and computationally to the positivity-preserving finite difference scheme on a fixed uniform mesh which has proven useful for solving the thin film problem.

Keywords: Adaptive mesh; Local refinement; Mixed finite element method; Thin film flow; Moving contact lines

1. Introduction

Adaptive procedures for the numerical solution of partial differential equations (PDEs), actively investigated since the late 1970s, are now standard tools in science and engineering – e.g. see [71] for references on adaptivity for elliptic PDEs. Adaptive finite element methods (FEMs) are a particularly meaningful approach for handling multiscale phenomena and making realistic computations feasible, especially on

* Corresponding author. Address: Department of Mathematics, Pennsylvania State University, University Park, State College, PA 16802, USA.
E-mail addresses: sun@math.psu.edu (P. Sun), rdr@math.sfu.ca (R.D. Russell), xu@math.psu.edu (J. Xu).

0021-9991/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcp.2006.11.005

irregular domains and in higher spatial dimensions with complex boundary conditions, where *a posteriori error estimators* are available as an essential ingredient of adaptivity. Such estimators are computable quantities depending on the computed solution(s) and data which provide information about the quality of approximation and may thus be used to make judicious mesh modifications. The ultimate purpose is to construct a sequence of meshes which will eventually equidistribute the approximation errors and, as a consequence, the computational effort. To this end, the a posteriori error estimators are split into element indicators which are then employed to make local mesh modifications by refinement (and sometimes coarsening). This naturally leads to loops of the form

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Refine}. \tag{1.1}$$

Starting from a coarse mesh, such an iteration has been widely successful in applications. Nevertheless, except for the rather complete description of the one-dimensional situation by Babuška and Rheinboldt [3], convergence of (1.1) in the multidimensional case is still largely an open issue. The fundamental paper [38] of Dörfler for the Poisson equation shows a linear error reduction rate for the energy norm towards a preassigned tolerance in finite steps. Recently Nochetto et al. [60] have constructed an adaptive FEM algorithm for elliptic PDEs and proved its linear convergence rate for the energy norm. Any prescribed error tolerance is thus achieved in a finite number of steps. It is in this context that there are some of the best current convergence results for adaptive finite element methods. All of the above a posteriori error estimators fall into a class of residual-type methods because they are all based on residual error on each element.

Besides the above developments for a posterior error estimators' convergence analysis, there is an alternative approach to produce a similar effect, viz., constructing (nearly) optimal meshes for suitable order piecewise finite element interpolation of a given function through a priori interpolation error estimators. Specifically, let $\Omega \in \mathscr{R}^n$ be a bounded domain, T a simplicial finite element mesh of $\overline{\Omega}$ with a fixed number N of elements, and u_I a piecewise finite element interpolation of a given function u defined on $\overline{\Omega}$. An optimal mesh could be obtained by minimizing the error $|||u - u_I|||$ in some sense, where the norm $|||\cdot|||$ is a classical *Sobolev* space norm.

This approach can be traced back to de Boor [34,33] where the problem of the best approximation by free knots splines was studied in one spatial dimension. In this work, the equidistribution principle was introduced specifically for computing equidistributed meshes. Actually the concept of equidistribution was first used by Burchard [21], and then by a number of researchers for studying grid adaptation. Pioneering work for adaptive finite element methods was done in [3] where a finite element mesh was shown nearly optimal in the sense of minimizing the H^1 norm error if the local errors are approximately equal for all elements. Thus, to get an optimal mesh, elements where the error is large are marked for refinement, while elements with a small error are left unchanged or coarsened.

Most of the adaptive finite element methods in the literature [4] are concerned with meshes that are shape-regular (which in two dimensions means that no element has a very small angle). This type of shape-regular finite elements is appropriate for physical problems that are fairly isotropic. But for many anisotropic problems (e.g. with sharp boundary layers or internal layers), the shape of elements can be further optimized, and an equidistribution of a scalar error density is not sufficient to ensure that a mesh is optimally efficient [32]. Nadler [62] studied the optimal triangulation for the discontinuous piecewise linear approximation for a quadratic function in the sense of minimizing error in the L^2 norm. For this optimal mesh, each triangle is equilateral under the Hessian metric, and the error is equidistributed on each triangle. The L^∞ case was studied by D'Azevedo and Simpson [30,31]. In [31] they developed a local linear interpolation error formula for quadratic functions. Based on this formula, D'Azevedo [30] obtained the same condition as Nadler's. Thereafter anisotropic mesh adaptation which aims to generate equilateral triangles under the metric induced by Hessian matrix was developed in [20,48,37] and successfully applied to computational fluid dynamic problems in two spatial dimensions [48,70].

Recently there have been some a priori interpolation error estimates for anisotropic finite elements [2,56,42]. Apel [2] obtained some estimates under a condition on the coordinate orientation and on the maximal allowable mesh angle. Formaggia and Perotto [42] exploited the spectral properties of the affine map from the reference triangle to the general triangle to get anisotropic estimates for the L^2 and H^1 interpolation error on linear finite elements in two dimensions. Kunert [56] introduced the matching function to measure the