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# Locally-corrected spectral methods and overdetermined elliptic systems

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**Abstract**

We present fast locally-corrected spectral methods for linear constant-coefficient elliptic systems of partial differential equations in $d$-dimensional periodic geometry. First, arbitrary second-order elliptic systems are converted to overdetermined first-order systems. Overdetermination preserves ellipticity, while first-order systems eliminate mixed derivatives, resolve convection–diffusion conflicts, and simplify derivative computations. Second, a periodic fundamental solution is derived by Fourier analysis and mollified for rapid convergence, independent of the regularity of the elliptic problem. Third, a new Ewald summation technique for first-order elliptic systems locally corrects the mollified solution to achieve high-order accuracy. We also discuss second-kind boundary integral equations based on single layer potentials formed with the mollified and corrected fundamental solution, which form a useful toolkit for solving general elliptic boundary value problems in general domains. The resulting spectral methods provide highly accurate solutions and derivatives for periodic problems.


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## 1. Introduction

A wide variety of time-independent physical problems find mathematical expression as second-order linear constant-coefficient elliptic systems

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{l=1}^{s} a_{ij}^{kl} u_{,ij}^{l} + \sum_{j=1}^{d}\sum_{l=1}^{s} b_{j}^{kl} u_{,j}^{l} + \sum_{l=1}^{s} c^{kl} u^{l} = f^{k}, \quad 1 \leqslant k \leqslant s, \tag{1}$$

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where $u^l_{,j}$ is the partial derivative of $u^l$ with respect to $x_j$. Such systems include the Poisson, Stokes and linear elasticity equations, which are often solved by specialized, inflexible codes for specific systems [1–3]. In this paper, we present a flexible new top-down approach which solves a wide spectrum of elliptic systems with uniform efficiency, and apply our new approach to develop accurate and efficient new spectral methods for elliptic problems in periodic domains. The new methods promise uniform accuracy for nonsmooth solutions and complex domains which are inaccessible to classical Fourier techniques.

Our approach converts any system (1) to a simple overdetermined first-order system

$$\sum_{j=1}^{d} A_j u_{,j} + A_0 u = f,$$

where each $A_j$ is a $p \times q$ matrix and $u$ is a $q$-vector. The conversion eliminates mixed derivatives, resolves convection-diffusion conflicts, and reduces condition numbers from $O(N^2)$ to $O(N)$ at resolution $N$. It solves all elliptic systems with a single efficient code, because linear algebra takes its proper place: correlating local relations between solution components. In the context of boundary integral formulations, the first-order conversion eliminates complicated relations between higher-order potential operators and employs single-layer potentials exclusively.

The paper is organized as follows. In Section 2, we convert arbitrary second-order elliptic problems to overdetermined first-order systems. In Section 3, we represent the solution to an overdetermined periodic first-order system as a "box potential" computed by integration against a fundamental solution. A periodic fundamental solution is derived by Fourier analysis in Section 4. A natural definition of ellipticity for first-order systems is justified. Suboptimal convergence of standard spectral methods for problems with nonsmooth solutions is discussed in Section 5. The classical Ewald summation technique which resolves convergence difficulties for the Poisson equation is reviewed in Section 6. In Section 7, a new Ewald summation technique for first-order elliptic systems is presented. It splits the fundamental solution into a global rapidly-converging Fourier series, mollified by a matrix exponential, and an error term. In classical Ewald summation, the error term is computed via special functions and integration, which cannot easily be done for a general elliptic system. Instead, we compute the error term by a simple Taylor expansion in Fourier space, which locally corrects the mollified fundamental solution by an asymptotic series of local differential operators. Our new mollification and local correction techniques are combined with the fast Fourier transform, Padé codes for small dense matrix exponentials, and high-order uncentered differencing to solve first-order elliptic systems in Section 8. In Section 9, we present a simple algebraic algorithm for the automatic computation of local correction coefficients which achieve high-order accuracy at minimal cost. Section 10 presents numerical experiments which verify efficiency and accuracy. In Section 11 we discuss extensions such as boundary integral equations for complex domains and variable-coefficient systems.

## 2. Conversion to first-order systems

Conversion to a first-order system replaces tiresome case-by-case analyses by linear algebra, computes derivatives of the solution automatically, and fosters the development of practical yet general codes for elliptic systems. In previous work on moving interfaces [4–6], for example, the various physical models of bulk processes require a wide array of solvers for elliptic and parabolic problems, and move the interface via computed normal derivatives of the solution. This complicated and sensitive technology would be greatly simplified by efficient codes for the stable computation of solutions and derivatives to general elliptic systems.

We convert the second-order system (1) to a first-order system by introducing all solution components $u^l$ and their first derivatives $u^l_{,j}$ as components of a $q$-vector $u = (u^1, u^2, \ldots, u^s, u^1_{,1}, u^2_{,1}, \ldots, u^s_{,d}) = (u_1, u_2, \ldots, u_q) \in \mathbb{R}^q$. The vector $u$ satisfies $p = (d + d(d-1)/2 + 1)s \geqslant q = (1 + d)s$ equations, which guarantee the following three conditions:

(a) the first $s$ components $(u_1, u_2, \ldots, u_s)$ constitute a solution to the original second-order elliptic system in the new variables