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A conservative semi-Lagrangian multi-tracer transport scheme (CSLAM) on the cubed-sphere grid ☆

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ABSTRACT

A conservative multi-tracer transport algorithm on the cubed-sphere based on the semi-Lagrangian approach (CSLAM) has been developed. The scheme relies on backward trajectories and the resulting upstream cells (polygons) are approximated with great-circle arcs. Biquadratic polynomial functions are used for approximating the density distribution in the cubed-sphere grid cells. The upstream surface integrals associated with the conservative semi-Lagrangian scheme are computed as line-integrals by employing the Gauss–Green theorem. The line-integrals are evaluated using a combination of exact integrals and high-order Gaussian quadrature. The upstream cell (trajectories) information and computation of weights of integrals can be reused for each additional tracer.

The CSLAM scheme is extensively tested with various standard benchmark test cases of solid-body rotation and deformational flow in both Cartesian and spherical geometry, and the results are compared with those of other published schemes. The CSLAM scheme is accurate, robust, and moreover, the edges and vertices of the cubed-sphere (discontinuities) do not affect the overall accuracy of the scheme. The CSLAM scheme exhibits excellent convergence properties and has an option for enforcing monotonicity. The advantages of introducing cross-terms in the fully two-dimensional biquadratic density distribution functions are also examined in the context of Cartesian as well as the cubed-sphere grid which has six local sub-domains with discontinuous edges and corners.

1. Introduction

The transport problem in computational fluid dynamics can either be cast in Lagrangian, Eulerian or in Arbitrary Lagrangian–Eulerian (ALE) form [1]. Lagrangian methods let the mesh travel and evolve with the fluid throughout the integration whereas Eulerian methods use a fixed mesh. Both methods have their strengths and weaknesses. The ALE method was developed in an attempt to combine the advantages of the Eulerian and the Lagrangian approaches by letting the mesh move in any prescribed manner as an extra independent degree of freedom. A popular choice of prescribed mesh movement is to run in Lagrangian mode for one time-step and then regrid (interpolate) back to the static and regular (Eulerian) mesh. In meteorological literature this approach is known as the semi-Lagrangian method [2]. A comprehensive review of conservative semi-Lagrangian methods are given in [3,4], and a stability analysis of these schemes is presented in [5].

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At every time-step, the semi-Lagrangian approach involves regridding (interpolating) quantities from a distorted Lagrangian mesh to a regular Eulerian mesh or vice versa, depending on the trajectories. Hence the transport problem is reduced to a regridding problem if the Lagrangian mesh movement is prescribed. For a variety of reasons it is desirable that the regridding procedure is conservative and monotonic. Conservative regridding is often referred to as remapping or rezoning. The problem of remapping quantities between arbitrary grids, which involves integration over overlapping areas between the grids, has received considerable attention in the literature due to its many applications. In general direct integration over arbitrary overlap areas is not practical. Through the pioneering work of Dukowicz [6,7] and Ramshaw [8] the remapping problem has been made practical by the application of Gauss–Green's theorem which converts area-integrals into line-integrals. This approach has been applied for up to second-order static grid-to-grid remapping in [9] and later the method was extended to third-order and optimized for the regular latitude–longitude and cubed-sphere grids [10].

In most atmosphere and ocean modeling applications the continuity equation must be solved multiple times for fluid density as well as dozens of tracers (chemical species). For example, the chemistry version of NCAR's Community Atmospheric Model (CAM) model [11] uses on the order of 100 prognostic tracers [12]. Therefore it is highly desirable that the numerical algorithm used for tracer transport is efficient and adaptable for a large number of tracers. In [13] an incremental remapping algorithm based on the semi-Lagrangian technique has been introduced for multi-tracer transport. Although incremental remapping has a high startup cost associated with geometry calculations, each additional tracer adds only a relatively small cost. The CSLAM algorithm considered herein follows this strategy.

Traditionally the regular latitude–longitude grid has been the preferred choice for global atmospheric models. However, models based on such grid system may have scalability issues. The scalability problems are either rooted from the non-scalable global numerical methods or the application of non-local polar filters. To address these problems, the atmospheric modeling community is developing numerical models based on more isotropic spherical grid systems that are free from singularities or contain weaker singularities. Also these grid geometries are amenable to *local* numerical methods such as the finite-volume method or element-based high-order Galerkin methods. The cubed-sphere geometry introduced by Sadourny [14] offers many computationally attractive features. Recently the cubed-sphere (spherical cube or expanded cube) geometry has been reintroduced in [15,16] with additional desirable features such as the equi-angular grid-spacing or orthogonality. Here we consider cubed-sphere grids based on the central (gnomonic) projection.

In this paper we optimize the more general method of Dukowicz [6,7] for transport on the cubed-sphere grid in two ways. Firstly, instead of using constant cell densities as in [6,8] or linear reconstructions of cell densities as in [7,9,17–19], we use the fully two-dimensional biquadratic reconstruction functions with a monotone option. Secondly, we exploit that for the gnomonic cubed-sphere grid it is possible to evaluate line-integrals along coordinate lines exactly [10]. Contrary to the incremental remapping algorithm, CSLAM is designed to allow for long time-steps with Courant numbers exceeding unity.

This paper is organized as follows. In Section 2 we introduce the CSLAM algorithm in Cartesian geometry. This involves defining the transport problem and introduce the notation required to mathematically describe the Lagrangian grid, in particular, the overlap regions between the static mesh and the Lagrangian grid. The conversion of area-integrals into line-integrals using Gauss–Green's theorem is described with details including the analytic integration of two-dimensional polynomial reconstruction functions. In Section 3, CSLAM is extended to the cubed-sphere geometry. Section 4 show results for standard test cases in Cartesian and spherical geometry. We will summarize the findings in Section 5.

2. Cartesian geometry

The two-dimensional transport equation for a tracer, in the absence of sources or sinks, can be written as

$$\frac{d}{dt}\int_{A(t)} \psi \, dA = 0, \tag{1}$$

(e.g., [13]) where ψ is the density (typically the product of the air density and the tracer concentration per unit mass), and the integration is over an arbitrary Lagrangian area $A(t)$ at time t, that is, an area that moves with the flow with no flux through its boundaries. A temporal discretization of (1) along the characteristics is

$$\int_{A(t+\Delta t)} \psi \, dA = \int_{A(t)} \psi \, dA, \tag{2}$$

where Δt is the time-step size.

In a semi-Lagrangian method either $A(t+\Delta t)$ or $A(t)$ is a static grid cell, or equivalently, either upstream (backward trajectories) or downstream (forward trajectories) cell tracking is used. Here we use the upstream approach so that $A(t+\Delta t)$ is a regular grid cell. Using the two-time level semi-Lagrangian terminology, $A(t+\Delta t)$ is referred to as the *arrival* (or Eulerian) cell and $A(t)$ the *departure* (or Lagrangian) cell.

In a two-dimensional Cartesian orthogonal grid system, let A_k be the kth (Eulerian) grid cell, where $k = 1, \ldots, N$, such that N is the total number of cells in the domain Ω. The departure cell corresponding to the arrival cell A_k is denoted by a_k (see Fig. 1). Note that there exists a one-to-one correspondence between departure and arrival cells such that the departure cells span Ω without gaps or overlaps between them,