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The constrained reinitialization equation for level set methods

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Based on the constrained reinitialization scheme [D. Hartmann, M. Meinke, W. Schröder, Differential equation based constrained reinitialization for level set methods, J. Comput. Phys. 227 (2008) 6821–6845] a new constrained reinitialization equation incorporating a forcing term is introduced. Two formulations for high-order constrained reinitialization (HCR) are presented combining the simplicity and generality of the original reinitialization equation [M. Sussman, P. Smereka, S. Osher, A level set approach for computing solutions to incompressible two-phase flow, J. Comput. Phys. 114 (1994) 146–159] in terms of high-order standard discretization and the accuracy of the constrained reinitialization scheme in terms of interface displacement. The novel HCR schemes represent simple extensions of standard implementations of the original reinitialization equation. The results evidence the significantly increased accuracy and robustness of the novel schemes.

1. Introduction

From their inception [1], level set methods have been established in many areas of and beyond computational physics to capture the motion of interfaces. This interface motion is in level set methods represented by an evolving scalar field. It is one of the great challenges of level set methods, however, that the scalar field often has to be frequently regularized to accurately compute discrete derivatives needed to evolve the field. Usually, the scalar field is initialized into a signed distance function and a reinitialization is performed to regularize the field. Other techniques to increase the fidelity of the level set method include the particle level set method introduced in [2] and coupled volume-of-fluid/level set methods discussed in [3,4], where the level set function is corrected using Lagrangian particles or the mass-conservation property of the volume-of-fluid method, respectively. Further methods rely on increased grid resolution using either adaptive mesh refinement as employed in [5] or a separately refined mesh for the level set method [6,7].

To reinitialize the level set field, a partial differential equation, which can be iteratively solved to transform an arbitrary scalar field into a signed distance function, is proposed in [8]. However, using this partial differential equation for the reinitialization is known to change the level set solution [9–11]. To be more precise, discretely solving this equation results in a significant displacement of the interface location, i.e., the interface is unphysically shifted during the reinitialization process. Modifications of the original formulation addressing this problem are proposed in [10,11,5] and more recently in [9], where the constrained reinitialization (CR) scheme is derived taking the location of the interface explicitly into account. The resulting CR scheme provides explicit equations to compute the signed distance function on the discrete computational points directly at the front, while away from the front the original reinitialization equation can be used. The method is second-order accurate in the location of the front and significantly reduces the displacement of the interface [9,12].

In this paper, the CR approach of Hartmann et al. [9] is generalized to higher-order schemes. Based on the expressions derived in [9], an explicit forcing term is formulated and introduced into the original reinitialization equation. This allows

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the reinitialization equation to be discretized with arbitrarily high-order schemes, while the forcing term acts as a constraint fixing the location of the interface.

The structure of the paper is as follows. After a brief introduction of the level set method in Section 2 and the reinitialization procedure in Section 3, the novel constrained reinitialization equation is introduced in Section 4. Results of two-and three-dimensional computations are given in Section 5, before the findings of the present paper are summarized in Section 6.

2. Level set transport

Consider an interface defined by

$$\phi_0 = \{(\mathbf{x},t) : \phi(\mathbf{x},t) = 0\}, \quad \mathbf{x} \in R^n,\ t \in R^+, \tag{1}$$

where $\phi(\mathbf{x},t)$ is the scalar level set function. For $n = 3$, let the components of the coordinate vector be denoted by $\mathbf{x} = (x,y,z)^T$. The level set function ϕ is specified as a signed distance function with respect to ϕ_0 with the properties

$$\begin{cases} \phi > 0 & \text{for } \mathbf{x} \in \Omega^+, \\ \phi = 0 & \text{for } \mathbf{x} \in \phi_0, \\ \phi < 0 & \text{for } \mathbf{x} \in \Omega^-, \end{cases} \tag{2}$$

where the computational domain Ω has been decomposed, $\Omega = \{\Omega^+, \Omega^-, \phi_0\}$, with $\Omega^+ \cap \Omega^- = \emptyset$ and $\phi_0 \notin \{\Omega^+, \Omega^-\}$. The level set equation governing the evolution of ϕ in Ω can be formulated

$$\partial_t \phi + \mathbf{f} \cdot \nabla\phi = 0, \tag{3}$$

where $\mathbf{f} = \mathbf{f}(\mathbf{x},t)$ is the extension velocity vector describing the motion of the local level set. A great advantage of level set methods is that geometric quantities such as the normal vector $\mathbf{n}$ and the curvature $\mathfrak{C}$ can be readily obtained from the scalar level set field

$$\mathbf{n} = -\frac{\nabla\phi}{|\nabla\phi|}, \tag{4a}$$

$$\mathfrak{C} = \nabla \cdot \mathbf{n}, \tag{4b}$$

where the normal vector $\mathbf{n}$ is defined such that it points into Ω^-.

3. Reinitialization

As noted in the introduction, the level set function ϕ is usually initialized into a signed distance function, which is the unique viscosity solution of the Eikonal equation

$$|\nabla\phi| = 1, \tag{5}$$

with the boundary condition $\phi = \phi_0$ at the interface. However, once initialized into such a signed distance function, the level set function ϕ usually does not retain this property under the evolution of Eq. (3) and needs to be reinitialized at regular time intervals [9,10,12]. Sussman et al. [8] reformulate the Eikonal equation (5) as an evolution equation in artificial time τ

$$\partial_\tau \phi^\nu + S(\tilde{\phi})(|\nabla\phi^\nu| - 1) = 0, \tag{6}$$

where the superscript $^\nu$ denotes the discrete pseudo-time step level. The quantity $S(\tilde{\phi})$ is a smoothed sign function of the perturbed level set function $\tilde{\phi} = \tilde{\phi}(\mathbf{x}, \tau = 0)$ being defined as

$$S(\tilde{\phi}) = \frac{\tilde{\phi}}{\sqrt{\tilde{\phi}^2 + \epsilon^2}}, \tag{7}$$

where ϵ is a smoothing parameter and is usually chosen equal to the grid spacing $\epsilon = h$ [8]. A modified smoothed sign function to achieve faster convergence in regions where $|\nabla\phi|$ is small is proposed by Peng et al. [10] and can be substituted for Eq. (7).

Analytically, Eq. (6) transforms an arbitrary scalar field ϕ into a signed distance function d with respect to the interface ϕ_0. However, when Eq. (6) is solved on a discrete grid the interface is significantly displaced during the reinitialization procedure [9,12]. The constrained reinitialization (CR) scheme developed recently by Hartmann et al. [9] addresses this problem by explicitly taking into account the location of the interface during the reinitialization process. The resulting expressions are explicit equations for the signed distance function on the cells at the interface, while on all other cells away from the interface Eq. (6) can be solved. For efficiency, a standard first-order upwind spatial discretization is used for Eq. (6) [9]. In [12], the novel scheme is compared against the standard reinitialization Eq. (6) spatially discretized with a first-order upwind and a fifth-order Hamilton–Jacobi WENO scheme [13]. The results show the CR scheme to be significantly superior