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Induced electric current-based formulation in computations of low magnetic Reynolds number magnetohydrodynamic flows

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ABSTRACT

We use the induced electric current as the main electromagnetic variable to compute low magnetic Reynolds number magnetohydrodynamic (MHD) flows. The equation for the induced electric current is derived by taking the curl of the induction equation and using Ampère's law. Boundary conditions on the induced electric current are derived at the interface between the liquid and the thin conducting wall by considering the current loop closing in the wall and the adjacent liquid. These boundary conditions at the liquid–solid interface include the Robin boundary condition for the wall-normal component of the current and an additional equation for the wall potential to compute the tangential current component. The suggested formulation (denominated *j*-formulation) is applied to three common types of MHD wall-bounded flows by implementing the finite-difference technique: (i) high Hartmann number fully developed flows in a rectangular duct with conducting walls; (ii) quasi-two-dimensional duct flow in the entry into a magnet; and (iii) flow past a magnetic obstacle. Comparisons have been performed against the traditional formulation based on the induced magnetic field (*B*-formulation), demonstrating very good agreement.

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1. Introduction

High performance computing aimed at modeling magnetohydrodynamic (MHD) phenomena is a powerful method for studying complex liquid metal (LM) flows under non-uniform strong magnetic fields in many practical applications related to fusion reactors and metallurgy. Currently, much consideration is given to the development of new models and computer codes for MHD flows in a strong magnetic field for both open [1] and closed [2] duct flows as applied to LM cooling of the so-called plasma facing components of a fusion-power reactor. The starting point in such computations is the mathematical model, whose selection is influenced by the choice of flow/electromagnetic variables. Ordinary flows of incompressible fluids can be modeled with the Navier–Stokes equations written in various forms [3] based on the usage of primitive variables, i.e. velocity and pressure ($\mathbf{V}, P$); as well as vorticity (ω) and the stream function (ψ) (in 2-D flows); and vorticity–velocity or vector potential–vorticity. In the case of MHD flows, the governing equations include an additional set of equations, derived from the Maxwell equations, which in turn can be formulated in different ways. We can refer to several formulations of considerable use that implement the electric scalar (φ) or magnetic vector ($\mathbf{A}$) potential, as well as the magnetic field ($\mathbf{B}$) [4]. Other formulations based on the current vector potential ($\mathbf{T}$) or some combinations of the above quantities, for instance $\mathbf{A} - \varphi$, are also in a general use but mostly applied to calculations of eddy currents in a solid conductor [5]. Attempts on

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implementation of a formulation making use of the induced electric current as the main electromagnetic variable are relatively recent and still limited to a few classes of MHD problems. The so-called velocity–current formulation, in which the induced electric current is used as the principal electromagnetic field variable and the electric potential plays the role of a Lagrange multiplier associated with the current density, was first introduced in [6] and further elaborated in [7,8] for a class of stationary MHD problems. As stressed in [6], the current-based formulation avoids some disadvantages specific to the traditional velocity–magnetic field formulation. The important point to stress here is that modeling ordinary or MHD flows involves the freedom to chose a proper set of dependent variables. For a particular MHD flow problem, selecting one or another set of electromagnetic variables in combination with a proper numerical technique can result in a higher accuracy and faster convergence. Vice versa an inappropriate choice can lead to unphysical results as well as poor or no convergence at all. The choice of the electromagnetic variables can also affect the size and the shape of the integration domain and the way in which the boundary conditions are formulated that ultimately affects the computation cost. In the present paper, we elaborate a formulation, which similarly to studies in [6–8] makes use of the induced electric current but unlike the variational approach introduced in the abovementioned studies, implements the finite-different technique. By doing this, a governing equation for the induced electric current is derived by taking the curl of the induction equation and using Ampère's law. The suggested formulation can be used in computations not only steady but also time-dependent MHD flows. Additionally, "thin conducting wall" boundary conditions for the induced electric current are derived assuming that the thickness of the flow-bounding solid wall is significantly smaller than the characteristic flow dimension so that the electric current entering the wall from the liquid flows in the wall almost tangentially. The suggested formulation does not require computations of the electric potential in the flow region but an additional equation for the wall potential has to be solved at the liquid–solid interface to compute the tangential component of the induced current at the interface. The wall-normal current component at the interface is described with the Robin boundary condition. In this way, the derived equation for the induced electric current, the boundary conditions for the tangential and wall-normal current components along with the flow equations and proper conditions on the velocity components form a closed problem.

Typically, LM MHD flows in various applications are characterized by three dimensionless parameters: the Hartmann number ($Ha = B_0L\sqrt{\sigma/\rho\nu}$), the Reynolds number ($Re = U_0L/\nu$), and the magnetic Reynolds number ($Re_m = \mu\sigma U_0L$). Here, B_0, L and U_0 are the characteristic magnetic field, flow dimension and velocity; while μ, σ, ρ and ν are the magnetic permeability, electrical conductivity, density and kinematic viscosity of the fluid, correspondingly. Hartmann number squared gives an estimate of the ratio of magnetic to viscous forces, while Re_m (when small) estimates the ratio of induced to applied magnetic field. In most applications involving LM MHD flows at industrial or laboratory scales, Re_m is much less than unity. Historically, the main challenge in modeling MHD wall-bounded flows is related to the need for fine resolution of MHD boundary layers that appear at the walls perpendicular to the applied magnetic field, known as Hartmann layers, whose thickness scales as $1/Ha$. In the fusion applications, the Hartmann number can be as high as 10^3–10^4, indicating the existence of very thin Hartmann layers. The small thickness of the Hartmann layer is, however, not the only limitation in advancing to higher Ha. In practice, computing high Hartmann number flows is also complicated by the fact that any local numerical error, for instance that in the Hartmann layers, will spread over the whole flow domain due to tight coupling between the flow and electromagnetic variables, distorting the computed flow field at a very short timescale. In the last two decades, many MHD computations were performed for duct flows (see e.g. [9–14]), but almost all of them are limited to $Ha \approx 10^2$. For example, in the widely cited study by Sterl [11], finite-volume 2-D and 3-D computations of MHD flows in rectangular ducts were conducted for both constant and space-varying magnetic fields with a maximum value of Ha of only 100 in the 3-D case. Such a limitation can be explained by the use of Ohm's law and the electric potential, which were implemented in all studies cited above. Namely, Ohm's law states that at low magnetic Reynolds numbers, the induced electric current $\mathbf{j}$ is given by

$$\mathbf{j} = \sigma(-\nabla\varphi + \mathbf{V} \times \mathbf{B}). \tag{1}$$

In typical LM MHD flows exhibiting Hartmann layers, the two terms on the right-hand side of Eq. (1) are of the same order, while the induced electric current is smaller by a factor of $1/Ha$. In this situation, small errors in computations of the electric potential or the velocity field may result in unacceptably high errors in $\mathbf{j}$, limiting the computations to relatively low Ha. In such situations (if special measures are not taken), the induced electric current path is not closed in the computational domain [12]. It appears that a numerical technique that does not fully compensate this kind of errors is impractical. Using higher accuracy schemes, finer meshes or double precision computations will probably extend the computations to higher values of Ha but will not solve the problem in principle. Fortunately, the problem of high Hartmann number computations can be mitigated in two ways. First, new numerical schemes, still based on the φ-formulation, can be developed, in which the error associated with Ohm's law is self-compensated. Such an approach has recently been demonstrated in [15] allowing accurate computations of MHD flows in a rectangular duct for Ha as high as 10^4. The second approach assumes abandoning the φ-formulation in favor of another formulation, which does not base the calculation of currents on Ohm's law. A well-known formulation of that type, which uses the magnetic field as the electromagnetic variable, involves Ampère's law to compute the electric current:

$$\mathbf{j} = \frac{1}{\mu}\nabla \times \mathbf{B}. \tag{2}$$

From the point of view of numerical computations, relation (2) has some advantage over (1). Namely, even with some inaccuracy in computing the magnetic induction field $\mathbf{B}$, the electric current calculated with (2), by virtue of the vector identity