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# Shock capturing with PDE-based artificial viscosity for DGFEM: Part I. Formulation

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ABSTRACT

Artificial viscosity can be combined with a higher-order discontinuous Galerkin finite element discretization to resolve a shock layer within a single cell. However, when a non-smooth artificial viscosity model is employed with an otherwise higher-order approximation, element-to-element variations induce oscillations in state gradients and pollute the downstream flow. To alleviate these difficulties, this work proposes a higher-order, state-based artificial viscosity with an associated governing partial differential equation (PDE). In the governing PDE, a shock indicator acts as a forcing term while grid-based diffusion is added to smooth the resulting artificial viscosity. When applied to heat transfer prediction on unstructured meshes in hypersonic flows, the PDE-based artificial viscosity is less susceptible to errors introduced by grid edges oblique to captured shocks and boundary layers, thereby enabling accurate heat transfer predictions.



## 1. Introduction

The focus of this work is shock capturing for higher-order discretizations with a particular emphasis on shock capturing for strong, steady shocks. A specific motivation is the simulation of hypersonic flows for the estimation of heat transfer loads. Unstructured meshes offer significant promise for computational aerothermodynamic analysis on complex geometries, but solution quality using unstructured meshes for current state-of-the art methods is far inferior to that of structured meshes [2,4,3,5,6,1]. The poor solution quality manifests itself even in symmetric, simplified test cases with poor prediction of peak heat transfer rates and asymmetric surface heat transfer distributions. The problem stems from the dependence of current shock capturing methods on the alignment of the mesh with the shocks. For simple problems, structured meshes can be designed to align the mesh with the shock such that numerical errors in shock capturing can be significantly reduced. However, for general meshes that do not align with the shock, the numerical errors at strong shocks can be significant and lead to non-physical variations that convect downstream to the boundary layer and corrupt surface heat transfer predictions.

Artificial viscosity, pioneered by von Neumann and Richtmyer [7], has been a common method of shock capturing in the context of streamwise upwind Petrov–Galerkin (SUPG) finite element methods, as proposed by Hughes et al. [8–11]. Researchers such as Hartmann and Houston [12,13] and Aliabadi et al. [14] have adopted this approach for use in discontinuous Galerkin methods as well, with good results, albeit only for $p = 1$ polynomial solutions.

The approach developed in this work is an extension of the sub-cell shock capturing method proposed by Persson and Peraire [15]. Specifically, Persson and Peraire introduced an elementwise-constant artificial viscosity that scales with the resolution length scale of a higher-order finite element method, $h/p$, such that the shock width is also $\mathcal{O}(h/p)$. Thus, for sufficiently high $p$, the shock can be captured within a single element. To locate the shocks in the flow field, Persson and Peraire

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developed a sensor based on the magnitude of the highest-order coefficients in an orthonormal representation of the solution.

As will be described in Section 3, an elementwise-constant artificial viscosity model has some inherent shortcomings. Specifically, element-to-element jumps in artificial viscosity lead to oscillations in state gradients that can corrupt the smoothness and accuracy of the downstream flow field. This research develops a smoother artificial viscosity by employing an artificial viscosity PDE model which is appended to the system of governing equations. Further, by combining the smooth PDE-based artificial viscosity with higher-order discretizations, the dependence of shock capturing on the grid orientation can be significantly reduced.

While the discretization used in this paper is the discontinuous Galerkin (DG) finite element method (FEM), the proposed shock-capturing approach is not strongly dependent on the particular higher-order method and will likely be equally effective for other discretizations in which sub-cell resolution is possible (SUPG/GLS, spectral volume, spectral difference, etc.). DG was first introduced by Reed and Hill [16] for the neutron transport equation. Much later, a foundation for DG methods applied to non-linear hyperbolic problems was established by Karniadakis et al. [17–25]. Independently, Allmaras and Giles [26,27] developed a second-order DG scheme for the Euler equations, building off of the work of van Leer [28–31]. Bassi and Rebay and Bey and Oden notably demonstrated the capabilities of DG for both the Euler and Navier–Stokes equations (including Reynolds Averaged Navier–Stokes) [32–36].

Section 2 reviews the DG FEM discretization of the compressible Navier–Stokes equations. Included in the review is the modification to the governing equations to append an artificial viscosity matrix for shock capturing. Section 3 motivates the use of a smooth, higher-order representation of artificial viscosity by highlighting the difficulties of a non-smooth formulation in one and two dimensions. Section 4 then presents the chief innovation of this research, a PDE for the control of artificial viscosity, and provides additional comparisons to a non-smooth formulation. Numerical results, including a hypersonic application of the new artificial viscosity model, are presented in Section 5.

## 2. Discretization

Let $\mathbf{u}(\mathbf{x},t) : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$ be the vector of $m$-state variables in $d$-dimensions for a general conservation law in the physical domain, $\Omega \subset \mathbb{R}^d$, given in the strong form by,

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{u}) - \nabla \cdot \mathcal{F}^v(\mathbf{u}, \nabla \mathbf{u}) = 0, \tag{1}$$

where $\mathcal{F}(\mathbf{u}) : \mathbb{R}^m \rightarrow \mathbb{R}^{m\times d}$ is the inviscid flux vector and $\mathcal{F}_v(\mathbf{u}, \nabla\mathbf{u}) : \mathbb{R}^m \times \mathbb{R}^{m\times d} \rightarrow \mathbb{R}^{m\times d}$ is the viscous flux.

### 2.1. Compressible Navier–Stokes equations

In the compressible Navier–Stokes equations, the conservative state vector is, $\mathbf{u} = [\rho, \rho v_i, \rho E]^T$, where $\rho$ is the density, $v_i$ is the velocity in the $i$th coordinate direction and $E$ is the total internal energy. The flux vectors are,

$$\mathcal{F}_i(\mathbf{u}) = \begin{bmatrix} \rho v_i \\ \rho v_i v_j + \delta_{ij} p \\ \rho v_i H \end{bmatrix}; \quad \mathcal{F}_i^v(\mathbf{u}, \nabla\mathbf{u}) = \begin{bmatrix} 0 \\ \tau_{ij} \\ v_j \tau_{ij} + \kappa_T \frac{\partial T}{\partial x_i} \end{bmatrix},$$

where $p$ is the static pressure, $H = E + p/\rho$ is the total enthalpy, $\delta_{ij}$ is the Kronecker delta, $\tau_{ij}$ is the shear stress defined below, $\kappa_T$ is the thermal conductivity, $T = p/\rho R$ is the temperature and $R$ is the gas constant. The pressure is related to the state vector by the equation of state, $p = (\gamma - 1)\rho(E - 0.5 v_i v_i)$, where $\gamma$ is the ratio of specific heats. The shear stress is, $\tau_{ij} = \mu(\partial v_i/\partial x_j + \partial v_j/\partial x_i) - \delta_{ij}\lambda \partial v_k/\partial x_k$, where $\mu$ is the dynamic viscosity and $\lambda = -\frac{2}{3}\mu$ is the bulk viscosity coefficient. Here the dynamic viscosity is assumed to adhere to Sutherland's Law, and the thermal conductivity is related to the viscosity by the Prandtl number, $Pr$. In the remainder of the paper, the following notation will be used for the viscous fluxes,

$$\mathcal{F}_i^v(\mathbf{u}, \nabla\mathbf{u}) = \mathbf{A}_{ij}^v(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_j},$$

where $\mathbf{A}_{ij}^v \in \mathbb{R}^{m\times m}$.

The discontinuous Galerkin finite element discretization proceeds by deriving a weak form of (1). The domain is subdivided by a triangulation, $\mathcal{T}_H$, into a set of non-overlapping elements, $\kappa$, such that $\Omega = \bigcup_{\kappa\in\mathcal{T}_H} \kappa$. Then, define $\mathcal{V}_H^p$, a vector-valued function space of discontinuous, piecewise-polynomials of degree $p$,

$$\mathcal{V}_H^p \equiv \{ v \in L^2(\Omega) | v|_\kappa \in P^p, \ \forall \kappa \in T_H \}.$$

The discontinuous Galerkin formulation is obtained by multiplying (1) by a test function, $\mathbf{v}_H \in (\mathcal{V}_H^p)^m$, integrating by parts, and accounting for the jumps from element-to-element by carefully defining the inviscid and viscous fluxes on element boundaries. The resulting DG formulation can then be expressed as the solution $\mathbf{u}_H(\cdot, t) \in (\mathcal{V}_H^p)^m$ to the semi-linear weighted residual (linear in the second argument), $\mathcal{R}(\mathbf{u}_H, \mathbf{v}_H) = 0, \ \forall \mathbf{v}_H \in (\mathcal{V}_H^p)^m$.