Journal of Computational Physics 228 (2009) 8856–8871

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Gaussian beam decomposition of high frequency wave fields

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ARTICLE INFO

Article history:
Received 21 April 2009
Received in revised form 29 August 2009
Accepted 31 August 2009
Available online 4 September 2009

Keywords:
Gaussian beams
High frequency waves
Asymptotic methods
Approximations

ABSTRACT

In this paper, we present a method of decomposing a highly oscillatory wave field into a sparse superposition of Gaussian beams. The goal is to extract the necessary parameters for a Gaussian beam superposition from this wave field, so that further evolution of the high frequency waves can be computed by the method of Gaussian beams. The methodology is described for $\mathbb{R}^d$ with numerical examples for $d = 2$. In the first example, a field generated by an interface reflection of Gaussian beams is decomposed into a superposition of Gaussian beams. The beam parameters are reconstructed to a very high accuracy. The data in the second example is not a superposition of a finite number of Gaussian beams. The wave field to be approximated is generated by a finite difference method for a geometry with two slits. The accuracy in the decomposition increases monotonically with the number of beams.

1. Introduction

We consider the wave equation for $x \in \mathbb{R}^d$,

$$\begin{aligned} &\Box u \equiv \partial_{tt} u - c(x)\Delta u = 0, \quad t > 0 \\ &u = f(x), \quad t = 0 \\ &\partial_t u = g(x), \quad t = 0. \end{aligned} \tag{1}$$

This equation is well posed in the energy norm,

$$\|u\|_E = \left(\int_{\mathbb{R}^d} \left[\frac{1}{c(x)} |u_t|^2 + |\nabla u|^2 \right] dx \right)^{1/2},$$

where ∇ is the gradient with respect to the spatial variables.

High frequency solutions to this equation are necessary in many scientific applications. While the equation has no scale, "high frequency" in this case means that there is a scale separation between the wave length and the domain of interest and that the sound speed $c(x)$ does not greatly vary on the scale of the oscillations. In such situations, direct discretization methods are notoriously computationally costly. To circumvent this, one often relies on asymptotically valid approximations such as geometric optics [1], geometrical theory of diffraction [2], and Gaussian beams [3–7].

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[1] Research partially supported by Chevron and the NSF (UT Austin RTG Grant No. DMS-0636586 and UCLA VIGRE Grant No. DMS-0502315).
[2] Research partially supported by Chevron.
[3] Research partially supported by Chevron and an Alfred P. Sloan Fellowship.

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2009.08.028

To set notation and remind the reader of the high frequency methods that this paper is focused on, we briefly review geometric optics and Gaussian beams. For a more detailed description of Gaussian beams with the similar notation, we refer the reader to [8,9]. In the high frequency limit with k the large high frequency parameter, one can look for special solutions of the wave equation that take the geometric optics form,

$$u(x,t) \sim a(x,t)e^{ik\phi(x,t)}. \tag{2}$$

Then, solving the wave equation is reduced to determining the amplitude function $a(x,t)$ and the phase function $\phi(x,t)$. Upon substituting (2) into the wave equation and collecting like powers of k, one obtains the eikonal equation for the phase and the transport equation for the amplitude,

$$|\phi_t|^2 - c(x)|\nabla\phi|^2 = 0$$
$$2\phi_t a_t - 2c(x)\nabla\phi\cdot\nabla a = -a\Box\phi.$$

In the method of geometric optics, these equations are solved by PDE techniques or by ODE ray tracing [1] for a real value phase and amplitude. Alternatively, in the Gaussian beams method, one relaxes the conditions on ϕ and a to allow them to take on complex values and one expresses them as Taylor polynomials about a characteristic ray, $(\mathcal{X}(s),\mathcal{T}(s))$, that originates at some point $(y,0)$ with ray parameter s:

$$\phi(x,t;y,s) = \Phi(s) + \Phi_t(s)(t-\mathcal{T}) + \nabla\Phi(s)\cdot(x-\mathcal{X}) + \frac{1}{2}[(x,t)-(\mathcal{X},\mathcal{T})]\cdot Hess[\Phi](s)[(\mathcal{X},\mathcal{T})-(x,t)] \tag{3}$$

$$a(x,t;y,s) = A(s). \tag{4}$$

Here, $Hess[\Phi]$ is Hessian matrix of Φ (which includes the second order x and t derivatives) and the above coefficients are defined through the ray tracing system of ODEs (using the shorthand notation $\tau = \Phi_t, \xi = \nabla\Phi, M = Hess[\Phi]$ and $\dot{} = \frac{d}{ds}$):

$$\dot{\mathcal{T}} = 2\tau$$
$$\dot{\mathcal{X}} = -2c(x)\xi$$
$$\dot{\tau} = 0$$
$$\dot{\xi} = |\xi|^2\nabla c$$
$$\dot{\Phi} = 0$$
$$\dot{M} = -MDM - MB - B^tM - C$$
$$\dot{A} = -A\Box\Phi$$

The matrices B, C, and D are $(d+1)\times(d+1)$ dimensional and defined as derivatives of $p(x,t,\xi,\tau) = |\tau|^2 - c(x)|\xi|^2$:

$$(B)_{kl} = \frac{\partial^2 p}{\partial\zeta_k\partial z_l}, \quad (C)_{kl} = \frac{\partial^2 p}{\partial z_k\partial z_l}, \quad (D)_{kl} = \frac{\partial^2 p}{\partial\zeta_k\partial\zeta_l},$$

with $z = (x,t)$ and $\zeta = (\xi,\tau)$. Thus defined, ϕ and a do not satisfy the eikonal and transport equation exactly, except on the ray; nonetheless, u given by Eq. (2) will be an asymptotic solution of the wave equation (see [8,9]).

To obtain a Gaussian beam solution, one has to determine the Taylor coefficients and the initial beam center y. Note that due to the relations between these coefficients that the eikonal equation (and its derivatives) provide, one only needs to determine the derivatives that involve x to determine all of the coefficients (up to the sign of ϕ_t). Also, although in the expression for ϕ and a both s and t appear as separate parameters, they are related through the condition $\mathcal{T}(s) = t$. What makes this type of construction give a valid asymptotic solution to the wave equation is that the x derivative block of the imaginary part of the Hessian matrix is a positive definite matrix. One can show that if this condition holds at $t = 0$, it will hold for all t, see [8]. This gives the name of the method, as at any given t the magnitude of the solution has a Gaussian shape.

Whether one uses geometric optics or Gaussian beams, an important fact to recognize is that the initial data for the wave equation, f and g in Eq. (1), have to fit with the special form of the solution. For geometric optics we need the initial data to top order in k to be of the form, $f(x) = a(x)\exp(ik\phi(x))$, with real valued phase ϕ, while for Gaussian beams, we need it to be $f(x) = a(x;y)\exp(ik\phi(x;y))$, where a and ϕ given by Taylor expansions about y. To see the required form for g, one recognizes that a and ϕ are functions of t as well and differentiates.

Finally, one can exploit the linear nature of the wave equation by finding the solution for N different initial data. Adding these together gives a solution to the wave equation with initial data given by the sum of their individual initial data. For the case of Gaussian beams, this means that the solution we can obtain has initial data of the form,

$$\sum_{n=1}^{N} a^n(x;y^n)e^{ik\phi^n(x;y^n)}.$$

In many applications, the available data is not typically in the form required for geometric optics or Gaussian beams. Thus we need to re-represent it in the appropriate from. A common method is to represent the field using the Fourier transform, so that it is in the form of an amplitude function times an exponential involving a phase [5,7]. This approach has some