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Integral equation methods for elliptic problems with boundary conditions of mixed type ☆

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ABSTRACT

Laplace's equation with mixed boundary conditions, that is, Dirichlet conditions on parts of the boundary and Neumann conditions on the remaining contiguous parts, is solved on an interior planar domain using an integral equation method. Rapid execution and high accuracy is obtained by combining equations which are of Fredholm's second kind with compact operators on almost the entire boundary with a recursive compressed inverse preconditioning technique. Then an elastic problem with mixed boundary conditions is formulated and solved in an analogous manner and with similar results. This opens up for the rapid and accurate solution of several elliptic problems of mixed type.

1. Introduction

The need to solve elliptic problems with different types of boundary conditions on different parts of a connected boundary often arises in computational physics. Elastic specimens partly held fixed and partly subjected to traction [2,18] and loaded composites with interface cracks [6,17] are common setups with mixed Dirichlet- and Neumann-type conditions. More generally, elliptic problems for multiphase materials where some continuity conditions hold on internal interfaces and loads are applied to a connected outer boundary belong to this class. Grain boundary diffusion in finite-size polycrystals [20,24] and coupled Stokes and Darcy flow [22] are two examples. Solvers based on integral equations, which are superior for pure boundary conditions, are not always applicable for mixed conditions. When they do apply and the conditions vary on a connected boundary, see [11] for an overview, they are often less advantageous than for pure boundary conditions.

It is hard to find integral equations for mixed problems that are of Fredholm's second kind with operators that are compact on the entire boundary. This is the essential difficulty when boundary conditions vary on contiguous boundary parts [23]. The second-kind-compact-operator property is what makes integral equation methods competitive. This property helps in retaining the condition number of the underlying mathematical problem throughout the solution process.

Using primitive functions of Neumann data, one can sometimes find integral equations for mixed planar problems that are singular with discontinuous coefficients in the sense of Section 116 of [15]. Such equations may require *reduction*, that is, the application of a pseudo-inverse to the dominant operator, for well-posedness. A great advantage with reduction

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is that it transforms a singular integral equation into one of Fredholm's second kind with operators that are compact on the entire boundary, provided the boundary is smooth. Reduction is certainly efficient when high accuracy is of interest [6,9]. Drawbacks include that it takes some effort to construct the pseudo-inverse and that it is hard to treat non-smooth boundaries [3] and boundary conditions that change type more than once. Mikhlin uses reduction to derive a second kind equation with compact operators for the mixed problem of the theory of elasticity, see Eq. (17) of Section 72 in [13]. This equation is not written out on explicit form and has, to our knowledge, never been used for numerics.

If one gives up the search for second kind equations with compact operators and is content with discretizing and solving just any integral equation, chiefly for the benefit of dimensionality reduction, the numerical results could suffer. Especially so in the vicinity of *singular boundary points*, that is, points where the boundary conditions change type and where the solution may have a complicated asymptotic behavior. Adaptive mesh refinement close to such points is often not a good idea since it can excite severe ill-conditioning. In general, any attempt at mesh refinement increases ill-conditioning in the absence of the second-kind-compact-operator property.

This paper takes a new approach to mixed boundary conditions. Like the classic works [13,15] we strive for integral equations that behave as if they were of Fredholm's second kind with compact operators everywhere. But while the classic works use reduction to achieve this, we use *recursive compressed inverse preconditioning* (a local multilevel technique developed to deal with weaker singularities stemming from boundary irregularities [8]). The advantages with trading reduction for recursive compressed inverse preconditioning are flexibility in modeling and simplicity in programming. Several types of complications can be treated within the same framework.

For brevity we only consider two problems: Laplace's equation in the plane, introduced in Section 3 and used to illustrate general ideas, and planar elasticity, chosen as to let these ideas work in a more challenging setting in Section 9. A key ingredient in the transition from Laplace's equation to elasticity is the particular choice of representation (55) and (56). Sections 2, 5–7 discuss quadrature techniques for non-smooth kernels and review recursive compressed inverse preconditioning in the present environment. The computational process is straight-forward, once these issues are settled, and Sections 8 and 10 present very accurate results.

2. Discretization and quadrature

We use Nyström discretization for the integral equations and composite 16-point Gauss–Legendre quadrature as our basic quadrature tool. To keep the notation short we make no distinction between points or vectors in a real plane $\mathbb{R}^2$ and points in a complex plane $\mathbb{C}$. All points will be denoted z or τ. Let Γ be the smooth boundary of a simply connected domain Ω and let Γ be given orientation. Let $\tau(t)$, $t_\alpha < t \leqslant t_\beta$, be a parameterization of Γ and let there be n_Λ quadrature panels Λ_j, $j = 1, \dots, n_\Lambda$, of approximately equal length placed on Γ. Then one can easily compute $N = 16n_\Lambda$ nodes t_j and weights $w_j, j = 1, \dots, N$, associated with integration in t. Let f be a layer density on Γ. The parameterization allows us to view f both as function of position $f(\tau)$ and of parameter $f(t)$. The argument indicates which view is taken in a particular situation. Differentiation with respect to parameter t is indicated with a prime. The abbreviations $\tau_j = \tau(t_j), f_j = f(t_j), \tau'_j = \tau'(t_j)$, and $f'_j = f'(t_j)$ are used.

We shall discretize several integral operators on Γ. If the integral kernel $K(\tau, z)$ and layer density $f(\tau)$ are piecewise smooth, the basic quadrature

$$\int_\Gamma f(\tau)K(\tau,\tau_j)\mathrm{d}\tau = \int_{t_\alpha}^{t_\beta} f(t)K(\tau(t),\tau_j)\tau'(t)\mathrm{d}t \approx \sum_{k=1}^{N} f_k K(\tau_k,\tau_j)\tau'_k w_k \tag{1}$$

should be accurate. If $K(\tau, z)$ is singular for $\tau = z$, special techniques are needed to retain high accuracy. This section reviews such techniques. Note that Nyström discretization of an integral equation means discretization of the integral operators at each quadrature point $\tau_j, j = 1, \dots, N$. The result of the discretization is a linear system with a square system matrix.

2.1. The Cauchy singular operator

We begin with the Cauchy singular integral operator

$$M_C f(\tau_j) = \frac{1}{\pi \mathrm{i}} \int_\Gamma \frac{f(\tau)\mathrm{d}\tau}{\tau - \tau_j}, \quad \tau_j \in \Gamma. \tag{2}$$

The integral is to be interpreted in the principal value sense. One option is to use *global regularization*

$$M_C f(\tau_j) = f_j + \frac{1}{\pi \mathrm{i}} \int_\Gamma \frac{(f(\tau) - f_j)\mathrm{d}\tau}{\tau - \tau_j}, \quad \tau_j \in \Gamma. \tag{3}$$

The integral has a continuous integrand when $f(\tau)$ is continuous. It can be discretized with basic quadrature and differentiation of $f(t)$ based on panelwise polynomial interpolation at the Legendre nodes.

A drawback with global regularization is that it may involve a fair amount of row summation for the diagonal elements of the system matrix. A cheaper alternative, in this respect, is *local regularization*