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Multigrid algorithms for high-order discontinuous Galerkin discretizations of the compressible Navier–Stokes equations

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ABSTRACT

Multigrid algorithms are developed for systems arising from high-order discontinuous Galerkin discretizations of the compressible Navier–Stokes equations on unstructured meshes. The algorithms are based on coupling both *p*- and *h*-multigrid (*ph*-multigrid) methods which are used in nonlinear or linear forms, and either directly as solvers or as preconditioners to a Newton–Krylov method.

The performance of the algorithms are examined in solving the laminar flow over an airfoil configuration. It is shown that the choice of the cycling strategy is crucial in achieving efficient and scalable solvers. For the multigrid solvers, while the order-independent convergence rate is obtained with a proper cycle type, the mesh-independent performance is achieved only if the coarsest problem is solved to a sufficient accuracy. On the other hand, the multigrid preconditioned Newton–GMRES solver appears to be insensitive to this condition and mesh-independent convergence is achieved under the desirable condition that the coarsest problem is solved using a fixed number of multigrid cycles regardless of the size of the problem.

It is concluded that the Newton–GMRES solver with the multigrid preconditioning yields the most efficient and robust algorithm among those studied.

1. Introduction

The inadequacy of current production-level computational fluid dynamics codes in delivering sufficient accuracy in numerical flow simulations, as well as in resolving a wide range of turbulence scales for reliable large eddy simulations have been widely realized over the past decade. Since these codes are typically based on low-order finite-volume methods, high-order methods such as discontinuous Galerkin (DG) methods have been advocated as alternative discretization techniques. DG methods are weighted residual methods with discontinuous approximate solution spaces typically consisting of polynomials of degree p defined on each element of the geometry triangulation. The inter-element connectivities are enforced through a proper definition of numerical fluxes along the shared boundaries between elements. High-order DG methods have advantages over continuous Galerkin methods in capturing features of convection-dominated flows, facilitating hp-adaptivity, ease of parallelization, and in the effectiveness of block-diagonal iterative solvers. For these advantages to be realized in industrial simulations, efficient solution strategies should be developed for systems arising from high-order DG discretizations. The importance of having efficient solution algorithms for DG methods is further appreciated when realizing that for a fixed mesh and a fixed approximation order (low to moderately high-orders), the DG discretization yields larger number of degrees of freedom compared to the continuous formulation.

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In this work, we consider the solution of the steady compressible Navier–Stokes equations using multigrid algorithms on unstructured (possibly anisotropic) meshes. Multigrid methods have been proved to be highly efficient in iteratively solving the system arising from implicit treatment of a wide range of physical problems including elliptic and hyperbolic equations [10,17,18,24].

For low-order discretizations (e.g. low-order finite-volume or element method), multigrid algorithms have been traditionally involved constructing a sequence of increasingly coarser meshes, and then using a fixed discretization technique on these coarser grids to form coarse level approximations (h-multigrid). One approach to obtain coarser meshes is the so-called agglomeration process involving merging together neighboring elements to obtain coarser grids. First-order accurate ($p = 0$) agglomeration multigrid methods for unstructured meshes are well established and deliver near optimal convergence rates (h-independence convergence) [8,9].

On the other hand, for high-order finite element methods, a natural choice for constructing coarse level approximations is to hold the computational mesh fixed and discretize the equations using a lower approximation order within each element (p-multigrid). This approach has been advocated and used in the context of spectral element methods for the last few decades [23,22]. More recently, two-level and p-multigrid algorithms have been studied for DG methods [29,16,13]. Persson and Peraire developed a two-grid preconditioner for the Newton–GMRES solution of the system arising from the high-order DG discretization of the compressible Navier–Stokes equations [29]. They considered the block Jacobi, block Gauss–Seidel and block ILU smoothers. The coarse level was either the $p = 0$ approximation or the $p = 1$, which was solved exactly using a direct solver. They concluded that the ILU preconditioner performs better than other preconditioners studied. Fidkowski et al. developed a p-multigrid algorithm for the high-order DG discretization of the compressible Navier–Stokes equations [13]. Their algorithm employed an element line Jacobi smoother in which lines of elements are formed using coupling based on the $p = 0$ discretization of the scalar convection–diffusion equation. They used a V-cycle multigrid with the coarsest level formed with the $p = 0$ approximation, and solved using a large number of smoothing iterations. Using the element line Jacobi smoother, they reported order-independent (p-independent) convergence rates, up to $p = 3$. However, they observed some h-dependence; that is, the convergence rate degrades as the mesh size grows.

The performance of the p-multigrid (and two-grid) solvers may degrade if the coarsest level problem is not solved efficiently. For solving the coarsest problem, although either a direct solver or a large number of smoothing iterations can be employed (as was employed in [29,13], respectively), they both yield poor asymptotic scaling with the mesh size. One remedy to this problem is to increase the total number of coarse levels through coupling both p- and h-multigrid (ph-multigrid) to more efficiently damp the large wave length error modes. This strategy was indeed used by Nastase and Mavriplis for the solution of Euler equations in two and three dimensions on unstructured triangular meshes [20,21] demonstrating p-independent and nearly h-independent convergence rates.

We herein extend the idea of coupling p- and h-multigrid algorithms to the high-order DG solution of the compressible Navier–Stokes equations using triangular meshes. One of the distinctive features of this work, compared to [20], is the investigation of two different cycling strategies for the ph-multigrid solution of the Navier–Stokes equations. Specifically, we show that unlike for the Euler equations, having the $p = 0$ approximation for the coarsest level does not yield optimal convergence for the Navier–Stokes equations, and we require the use of the $p = 1$ approximation as the coarsest level problem. We suggest solving the $p = 1$ problem itself using several cycles of a V-cycle ph-multigrid scheme. We also investigate the use of the ph-multigrid algorithm as a preconditioner within the context of a Newton–GMRES solver.

The remainder of the paper is organized as follows. In the next section, we introduce the steady compressible Navier–Stokes equations, before describing the DG discretization in Section 3. We then present the single-grid solver and the proposed multigrid algorithms and the preconditioned Newton–GMRES solver in Sections 4–6, respectively. In the final two sections, we present the numerical results and conclude the work.

2. Compressible Navier–Stokes equations

The steady compressible Navier–Stokes equations are written in vector form as

$$\frac{\partial}{\partial x_i}\left(\mathbf{f}_i^c(\mathbf{u}) - \mathbf{f}_i^v(\mathbf{u}, \nabla\mathbf{u})\right) = 0 \text{ in } \Omega (i = 1, \ldots, d), \tag{1}$$

where Ω is a bounded domain in d space dimensions with $d = 2$ or 3. The conservative state vectors are $\mathbf{u} \equiv [u_1, \ldots, u_{d+2}]^T = [\rho, \rho v_1, \ldots, \rho v_d, \rho E]^T$ with $\rho, \mathbf{v} = [v_1, \ldots, v_d]$ and E representing the density, velocity vector and total energy, respectively.

The convective and viscous fluxes are, respectively, defined as

$$\mathbf{f}_i^c(\mathbf{u}) = \begin{bmatrix} \rho v_i \\ \rho v_1 v_i + p\delta_{1i} \\ \vdots \\ \rho v_d v_i + p\delta_{di} \\ \rho H v_i \end{bmatrix}, \quad \mathbf{f}_i^v(\mathbf{u}, \nabla\mathbf{u}) = \begin{bmatrix} 0 \\ \tau_{1i} \\ \vdots \\ \tau_{di} \\ \tau_{ij} v_j + \mathcal{K}\frac{\partial T}{\partial x_i} \end{bmatrix}, \quad (i = 1, \ldots, d), \tag{2}$$