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A stochastic variational multiscale method for diffusion in heterogeneous random media

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Abstract

A stochastic variational multiscale method with explicit subgrid modelling is provided for numerical solution of stochastic elliptic equations that arise while modelling diffusion in heterogeneous random media. The exact solution of the governing equations is split into two components: a coarse-scale solution that can be captured on a coarse mesh and a subgrid solution. A localized computational model for the subgrid solution is derived for a generalized trapezoidal time integration rule for the coarse-scale solution. The coarse-scale solution is then obtained by solving a modified coarse formulation that takes into account the subgrid model. The generalized polynomial chaos method combined with the finite element technique is used for the solution of equations resulting from the coarse formulation and subgrid models. Finally, various numerical examples are considered for evaluating the method.

Keywords: Variational multiscale; Multiscale finite element; Operator upscaling; Stochastic diffusion; Subgrid modelling; Generalized polynomial chaos

1. Introduction

In this paper, we address a variational multiscale (operator upscaling) framework for numerical simulation of transient diffusion in a random medium whose diffusion coefficient is characterized by the presence of features at multiple length scales. The potential applications for the problem include heat transfer in composites [1,2] and flow in porous media [3], wherein, spatial variation in material properties requires a statistical description owing to gappy data and assumptions in constitutive models. Fully-resolved transient computations require spatial and temporal discretizations that can resolve the smallest length scales in the material data (here, the diffusion coefficient) and time scales in the solution, respectively. However, in practice, a coarse-scale description of the solution is deemed adequate. It is thus desirable to develop computational

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techniques that "solve for a coarse-scale solution by defining an appropriate coarse-scale problem that captures the effect of the fine-scales" [4]. This forms the backbone of most upscaling methods (see [5] for a comprehensive review).

Current research on upscaling techniques (more specifically, multiscale methods) aims at the derivation of a coarse-scale formulation that has the following characteristics: (i) it contains adequate fine-scale information, i.e. the fine-scale solution can be reconstructed from the coarse solution and subgrid results, (ii) the computation cost for the coarse-scale problem should scale sub-linearly with respect to the fully-resolved computation, and (iii) it should avoid assumptions of special problem structures like periodicity and scale separation. However, when these structures are present, we should be able to exploit them for increasing computation speed.

The most popular techniques developed for upscaling in deterministic context fall under the category of multiscale methods viz. the variational multiscale *VMS* method (also known as operator upscaling) [6–8], the heterogenous multiscale method [9,10], and the multiscale finite element method [11–13]. These methods typically involve introduction of multiscale basis functions at the coarse-scale. These multiscale basis functions include information about the fine-scale heterogeneities in the problem. Further related techniques involve the generalized finite element method [14] and the residual-free bubbles [15].

Parallel to the above developments in deterministic upscaling techniques, there have been considerable advances in computationally efficient stochastic analysis approaches. Traditionally, stochastic analysis techniques for partial differential equations (PDEs) involve Monte-Carlo sampling [16], perturbation analysis [17] and Neumann expansions [18]. However, these methods are limited either by the prohibitive computation cost, inability to handle large fluctuations and nonlinearity and/or complexity involved in derivation of these techniques (e.g. deriving perturbation methods for analyzing higher-order solution statistics becomes increasingly complex). These shortcomings are alleviated by using the generalized polynomial chaos expansion (GPCE) approach [19], a technique for representing stochastic fields with finite variance using Wiener–Askey hypergeometric polynomials [20,21]. The GPCE is a significant advancement on the spectral stochastic method [22] that uses Hermite polynomials for representing Gaussian and allied stochastic processes (e.g. log-normal process as an exponential of a Gaussian process). The GPCE approach has been used successfully in the context of fluid-flow [23,24], fluid-structure interaction [25] and natural convection [24]. A survey of numerical challenges in the implementation of the GPCE approach for the solution of stochastic PDEs is provided in [26].

Using deterministic upscaling techniques for statistical analysis of the solution typically involves the use of computationally expensive Monte-Carlo methods. Recently, there has been a considerable interest in stochastic homogenization, upscaling methods [27,28]. These methods however employ restrictive assumptions on the problem structure viz. periodicity and scale separation. However, considering the advances in deterministic upscaling methods and stochastic analysis approaches, it is time to address direct incorporation of the inherent randomness in material data and the effect of modelling assumptions in the design of upscaling methods. In this paper, we combine the variational multiscale VMS method [29,30], the multiscale finite element method and deterministic operator upscaling technique [1,3,4,6,11] and the GPCE/polynomial chaos approach [19,22] to derive a variationally consistent upscaling technique for the stochastic transient diffusion equation. The authors emphasize that though all of the above techniques are not a part of their original work, this paper introduces a framework to combine these techniques to derive a stochastic upscaling technique. Also, the paper addresses operator upscaling in the context of a transient multiscale diffusion equation, which to the best of the authors' knowledge is a novel contribution. Since randomness is effectively seen as an additional dimension in the problem [31], our method essentially performs upscaling for a class of problems corresponding to various realizations of the random material data (here, the diffusion coefficient).

The choice of VMS as the upscaling method and GPCE as the stochastic analysis method is motivated by a number of reasons. VMS and operator upscaling methods have emerged in recent years as computational paradigms for development of multiscale analysis [6,29]. The VMS approach essentially involves splitting the variational formulation for the governing equations into a coarse and a fine-scale part. The fine-scale part is then solved approximately to obtain the fine-scale solution model, that is substituted in the coarse-scale part of the variational formulation to obtain an upscaled problem. The authors have contributed to the development of the stochastic VMS method, wherein, algebraic models are used for the fine-scales and the GPCE