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Fourier spectral embedded boundary solution of the Poisson's and Laplace equations with Dirichlet boundary conditions

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ABSTRACT

A Fourier spectral embedded boundary method, for solution of the Poisson's equation with Dirichlet boundary conditions and arbitrary forcing functions (including zero forcing function), is presented in this paper. This iterative method begins by transformation of the Dirichlet boundary conditions from the physical boundaries to some corresponding regular grid points (which are called the numerical boundaries), using a second order interpolation method. Then the transformed boundary conditions and the forcing function are extended to a square, smoothly and periodically, via multiplying them by some suitable error functions. Instead of direct solution of the resulting extended Poisson's problem, it is suggested to define and solve an equivalent transient diffusion problem on the regular domain, until achievement of the steady solution (which is considered as the solution of the original problem). Without need of any numerical time integration method, time advancement of the solution is obtained directly, from the exact solution of the transient problem in the Fourier space. Consequently, timestep sizes can be chosen without stability limitations, which it means higher rates of convergence in comparison with the classical relaxation methods. The method is presented in details for one- and two-dimensional problems, and a new emerged phenomenon (which is called the saturation state) is illustrated both in the physical and spectral spaces. The numerical experiments have been performed on the one- and two-dimensional irregular domains to show the accuracy of the method and its superiority (from the rate of convergence viewpoint) to the other classical relaxation methods. Capability of the method, in dealing with complex geometries, and in presence of discontinuity at the boundaries, has been shown via some numerical experiments on a four-leaf shape geometry.

1. Introduction

In the embedded boundary methods (EBM), a domain Ω with irregular boundary $\partial\Omega$ (where the solution is sought on it), is surrounded with a bigger domain D with regular boundaries (coinciding with the considered coordinate axes). Then the problem is extended from the irregular domain Ω to the whole of the regular domain D, and solution of the extended problem is obtained by the use of high efficiency numerical methods. A typical drawing for a two-dimensional Cartesian grid is shown in Fig. 1.

Although suggestion of such methods has a long history which returns to the early days of numerical methods [7], just recently (because of some deficiencies of the conventional structured and unstructured grids in dealing with the moving boundary and multi-body problems, multiphase flows and so forth), these methods have become in the center of attention of many academic, as well as applied researchers [2,10,16,12,18].

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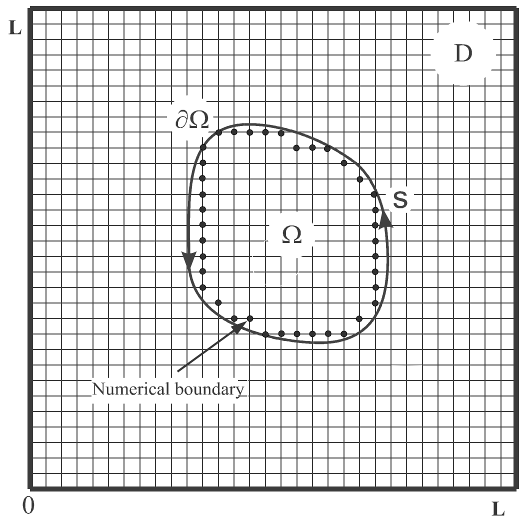

Fig. 1. Regular and irregular (physical) boundaries and the corresponding numerical boundary (generated for a typical uniform Cartesian grid), for a two-dimensional problem.

In this context, the fast Fourier transform-based (FFT-based) methods, which formally are $\mathcal{O}(N \log N)$, can be considered as the efficient methods for solution of the extended problems. Nevertheless, use of these methods in the EBM have been considered just recently, mainly because of their drawbacks in facing with discontinuities and their intrinsic restrictions in implementation of the general boundary conditions [2,4,6]. Prior to these works almost all of the spectral methods, suggested for the general irregular boundaries, employed a kind of the spectral element methods, or the boundary element methods in implementation of the Dirichlet boundary conditions [1,8].

The FFT-based embedded boundary solution of the Poisson's equation with homogeneous Neumann boundary condition and without any boundary condition (by appropriate extension of the forcing function), have been proposed in [2,4,6], respectively. The present article, as a generalization to these methods, is devoted to the Fourier embedded boundary solution of the Poisson's problem with Dirichlet boundary conditions.

In addition to their wide applications in the diffusion and filtering problems and the domain decomposition methods, these problems have a crucial role in the fluid flow simulations. In fact, the present method can be used directly in solution of the vorticity-stream function formulation of the two-dimensional incompressible Navier–Stokes equations [15,18]. Here, without loss of generality, only the one- and two-dimensional problems are discussed. But, as it will be seen, extension of the method to three-dimensional problems is straightforward. Moreover, the method is directly applicable to the Laplace equation (i.e. the Poisson's equation with zero forcing function), which is not the case for many other similar methods.

For a d-dimensional domain $\Omega \subset \mathbb{R}^d$ and its boundary $\partial\Omega$, and for the given functions $F : \Omega \to \mathbb{R}$ and $v_b : \partial\Omega \to \mathbb{R}$, the Poisson's problem with Dirichlet boundary condition is defined as

$$\sum_{i=1}^{d} \frac{\partial^2 v}{\partial x_i^2} = F \quad \text{in } \Omega, \tag{1}$$

$$v(\mathbf{S}) = v_s \quad \text{on } \partial\Omega. \tag{2}$$

In Fig. 1, the Ω has been shown as a two-dimensional domain with irregular closed boundary $\partial\Omega$, defined by a vector function $\mathbf{S}$. To stress their differences, the irregular boundary $\partial\Omega$ will be noted as the 'physical boundary', while the corresponding points on the uniform grid (marked by the bold circles in the figure), will be called the 'numerical boundary'. As a necessary condition for good representation of the physical boundaries, it is assumed that the vector function $\mathbf{S}$ is defined everywhere on the boundary, with a resolution sufficiently greater than the uniform computational grid.

Inspired by the methods of Bueno-Orovio et al. [6], Bueno-Orovio and Perez-Garcia [5], Boyd [2] and then Bueno-Orovio [4], the main idea is suitable extension of the forcing function F into $D \setminus (\Omega \cup \partial\Omega)$, such that the extended function can be transformed to the Fourier space. Then the solution of the extended problem will be found in the Fourier space, and finally the desired solution of the original problem (defined in Ω) can simply be found by ignoring the solution in the $D \setminus (\Omega \cup \partial\Omega)$.

Unfortunately, the Poisson's problem with Dirichlet boundary conditions in the above formulation, cannot be solved directly by the methods of Boyd [2], Bueno-Orovio [4] or Bueno-Orovio et al. [6], at least for the following two reasons:

(1) Absence of an explicit method for implementation of the Dirichlet boundary conditions in these works. However, addition of a homogeneous solution (which can be obtained from the boundary element method or capacity matrix