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Sixth order compact scheme combined with multigrid method and extrapolation technique for 2D poisson equation ☆

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ABSTRACT

We develop a sixth order finite difference discretization strategy to solve the two dimensional Poisson equation, which is based on the fourth order compact discretization, multigrid method, Richardson extrapolation technique, and an operator based interpolation scheme. We use multigrid V-Cycle procedure to build our multiscale multigrid algorithm, which is similar to the full multigrid method (FMG). The multigrid computation yields fourth order accurate solution on both the fine grid and the coarse grid. A sixth order accurate coarse grid solution is computed by using the Richardson extrapolation technique. Then we apply our operator based interpolation scheme to compute sixth order accurate solution on the fine grid. Numerical experiments are conducted to show the solution accuracy and the computational efficiency of our new method, compared to Sun–Zhang's sixth order Richardson extrapolation compact (REC) discretization strategy using Alternating Direction Implicit (ADI) method and the standard fourth order compact difference (FOC) scheme using a multigrid method.

1. Introduction

Poisson equation is a partial differential equation (PDE) with broad applications in mechanical engineering, theoretical physics and other fields. The two dimensional (2D) Poisson equation can be written in the form of

$$u_{xx}(x,y) + u_{yy}(x,y) = f(x,y), \quad (x,y) \in \Omega, \tag{1}$$

where Ω is a rectangular domain, or a union of rectangular domains, with suitable boundary conditions defined on $\partial\Omega$. The solution $u(x,y)$ and the forcing function $f(x,y)$ are assumed to be sufficiently smooth and have the necessary continuous partial derivatives up to certain orders.

A second order accurate solution can be computed by applying the standard second order central difference operators to $u_{xx}(x,y)$ and $u_{yy}(x,y)$ in Eq. (1). Higher order (more than two) accurate discretization methods need more complex procedure than the second order accurate discretization method to compute the coefficient matrix, but they usually generate linear

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systems of much smaller size, compared with that from the lower order accurate discretization methods [1,7,10]. There has been growing interest in developing higher order accurate discretization methods, especially the high order compact difference schemes, to solve partial differential equations (PDEs) [11,15,18,23,25,26]. We call them "compact" because these schemes only use the minimum three grid points in one dimension in the discretization formulas.

Previously, Chu and Fan [5,6] proposed a three point combined compact difference (CCD) scheme for solving two dimensional Stommel Ocean model, which is a special two dimensional convection–diffusion equation. They used Hermitian polynomial approximation to achieve sixth order accuracy for the inner grid points and fifth order accuracy for the boundary grid points. The advantage of the CCD scheme is that it can be used to solve many types of PDEs without major modifications. And the Alternating Direction Implicit (ADI) [14] method can be used to reduce the higher dimensional problems to a series of lower dimensional problems. So, their scheme is referred to as the implicit high order compact scheme because they do not compute the solution of the dependent variables of the PDEs directly. Instead, the first derivative and the second derivative of the dependent variables are computed at the same time.

In contrary, the explicit fourth order compact schemes [9,10,12,13,18] compute the solution of the variables directly, no redundant computation is needed. Some accelerating iterative methods like multigrid method and preconditioned iterative method have been used to efficiently solve the resulting sparse linear systems arising from the high order compact finite difference discretizations [22,24,25]. But the higher order explicit compact schemes are more complicated to develop in higher dimensions [8,27], compared with the implicit compact schemes. As far as we know, there is no existing explicit compact scheme on a single scale grid that is higher than the fourth order accuracy.

Since a sixth order explicit compact scheme may be impossible to develop on a single scale grid, the multiscale grid method has been considered to achieve the sixth order accuracy for the explicit compact formulations. Sun and Zhang [20] first proposed a sixth order explicit finite difference discretization strategy for solving the 2D convection–diffusion equation. They used ADI method to compute the fourth order accurate solution on the fine and the coarse grids first, then apply the Richardson extrapolation technique and an operator based interpolation scheme in each ADI iteration to achieve the sixth order accurate solution on the fine grid. The major disadvantage of Sun–Zhang's method is that the ADI iteration is not scalable with respect to the meshsize. When the mesh becomes finer, the number of ADI iterations needed for convergence increases quickly.

By using the idea of two scale grid computation from Sun–Zhang's method, we intend to develop a new explicit sixth order compact computing strategy for the 2D Poisson equation, which can efficiently solve the resulting linear system and is scalable with respect to the problem size. We do not use the ADI method, instead, we develop a multigrid method that is similar to the full multigrid method as our convergence acceleration method. With point Gauss–Seidel relaxation method and line Gauss–Seidel relaxation method, we iteratively solve the resulting sparse linear system to get the fourth order accurate solutions on both the fine and the coarse grids. Then we apply the Richardson extrapolation technique combined with our new operator based interpolation scheme to compute the sixth order accurate solution on the fine grid.

In this paper, we present the sixth order compact difference discretization strategy for the 2D Poisson equation in Section 2. In Section 3, we develop our modified multigrid method to solve the fourth order accurate solution on the fine and the coarse grids. Section 4 contains the numerical experiments to demonstrate the high accuracy of the sixth order compact difference scheme, as well as the computational efficiency of our modified multigrid method. Concluding remarks are given in Section 5.

2. Sixth order compact approximations

Our explicit sixth order compact difference scheme is based on the fourth order compact discretization on the two scale grids. In this section, we first introduce the fourth order compact difference scheme for the 2D Poisson equation. The basic idea is from Zhang's previous papers [20,25,28]. More detailed discussions about the fourth order compact difference schemes can be found in [9,17].

In order to discretize Eq. (1), let us consider a rectangular domain $\Omega = [0, L_x] \times [0, L_y]$. We discretize Ω with uniform meshsizes $\Delta x = L_x/N_x$ and $\Delta y = L_y/N_y$ in the x and y coordinate directions, respectively. Here N_x and N_y are the number of uniform intervals in the x and y coordinate directions. The mesh points are (x_i, y_j) with $x_i = i\Delta x$ and $y_j = j\Delta y$, $0 \leqslant i \leqslant N_x$, $0 \leqslant j \leqslant N_y$.

We write the standard second order central difference operators as

$$\delta_x^2 u_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}, \quad \delta_y^2 u_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}.$$

Using Taylor series expansions, at the grid point (x_i, y_j), we have

$$\delta_x^2 u_{i,j} = u_{xx} + \frac{\Delta x^2}{12} u_x^4 + \frac{\Delta x^4}{360} u_x^6 + O(\Delta x^6), \tag{2}$$

and

$$\delta_y^2 u_{i,j} = u_{yy} + \frac{\Delta y^2}{12} u_y^4 + \frac{\Delta y^4}{360} u_y^6 + O(\Delta y^6). \tag{3}$$