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Towards a compact high-order method for non-linear hyperbolic systems, II. The Hermite-HLLC scheme

G. Capdeville *

Département de Mécanique des Fluides, Ecole Centrale de Nantes, 1, Rue de la Noe, B.P. 92101, 44321 Nantes cedex 3, France

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ABSTRACT

In a finite-volume framework, we develop an approximate HLL Riemann solver specific to weakly hyperbolic systems. Those systems are obtained by considering not only the variable but also its first spatial derivative, as unknowns. To this aim, we rely upon the theory of "δ-shock waves", newly developed in the scalar case.

First, we demonstrate that the extended version of the HLLE scheme to weakly hyperbolic systems is compatible with the existence of Dirac measures in the solution. Then, we develop a specific Hermite Least-Square (HLSM) interpolation that enables to generate a high-order and compact scheme, without creating spurious oscillations in the reconstruction of the variable or its first derivative. Extensive numerical experiments make it possible to validate the method and to check convergence to entropy solutions.

Relying upon those results, we construct a new HLL Riemann solver, suited for the extended one-dimensional Euler equations. For this purpose, we introduce the contribution of a contact discontinuity inside the definition of the solver. By using a formal analogy with the scalar study, we demonstrate that this solver tolerates the existence of "δ-shock waves" in the solution. Numerical experiments that follow help to validate some of the assumptions made to generate this scheme.

1. Introduction

In this paper, we develop a Riemann solver for a class of coupled hyperbolic systems of conservation laws. For that purpose, we rely upon the newly developed theory of "δ-shock waves".

Let us briefly recall this theory and its main results.

1.1. δ-Shock wave type solutions [1–6]

For some cases of coupled hyperbolic systems, "non-classical solutions" may occur when the Riemann problem does not possess a weak solution except for some particular initial data. In order to solve Cauchy problems in these situations, it becomes necessary to introduce new singularities called "δ-shocks", which are solution of the coupled hyperbolic systems.

To illustrate this problem, let us consider the following scalar non-linear conservation law:

$$u_t + f(u)_x = 0 \tag{1}$$

where, $f(u)$, is a convex smooth function, $f'(u) > 0$.

* Tel.: +33 2 4037 1651; fax: +33 2 4037 2523.
E-mail address: guy.capdeville@ec-nantes.fr

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By differentiating this equation with respect to x and denoting $v \equiv u_x$, we obtain the following coupled hyperbolic system:

$$\begin{cases} u_t + f(u)_x = 0 \\ v_t + (f'(u)v)_x = 0 \end{cases} \tag{2}$$

System (2) is extremely degenerate – sometimes it is called "weakly hyperbolic" – with repeated eigenvalue $\lambda \equiv f'(u)$ and repeated eigenvector $(0,1)^t$. As a consequence, the linear component, v, of the solution, may contain Dirac measures while the non-linear component, u, preserves bounded variations.

If we consider the Cauchy problem for system (2), with the initial data

$$(u(x,t=0), v(x,t=0)) = \begin{cases} (u^l, v^l) & x < 0 \\ (u^r, v^r) & x > 0 \end{cases} \tag{3}$$

where $(u, v)^{l,\ r}$ are given constants, one can demonstrate that there are only two kinds of solutions for such a problem. The first one involves "δ-shock" waves, the second involves vacuums.

According to [5,7], the following theorem holds:

(a) if $u^l \geqslant u^r$, it exists a unique weak solution to the Cauchy problem (2) and (3). This solution has the following form:

$$\begin{cases} u(x,t) = u^l + [u] \times H(-x + x_s(t)) \\ v(x,t) = v^l + [v] \times H(-x + x_s(t)) - e(t) \times \delta(-x + x_s(t)) \end{cases} \tag{4}$$

Moreover, this solution satisfies the entropy condition

$$f'(u^l) \leqslant \frac{\mathrm{d}x_s}{\mathrm{d}t} \leqslant f'(u^r) \tag{5}$$

where we denoted: $[\cdot] \equiv (\cdot)^r - (\cdot)^l$, the jump of the variable u or v across the discontinuity. $H(x)$ is the Heaviside function and $\delta(x)$ is the delta function. In addition, functions $x_s(t)$ and $e(t)$ are defined by the system

$$\begin{cases} \frac{\mathrm{d}x_s}{\mathrm{d}t} = \frac{[f(u)]}{[u]}\Big|_{x=x_s(t)} \\ \frac{\mathrm{d}e}{\mathrm{d}t} = \left([f'(u)v] - [v]\frac{[f(u)]}{[u]}\right)\Big|_{x=x_s(t)} \end{cases} \tag{6}$$

with the initial data determined by (3) and $x_s(0) = 0$.

The solution (4) that satisfies (5) and (6), is called a "δ-shock" wave solution of the Cauchy problem (2) and (3). In this solution, the v component contains a δ measure while the u component is piecewise constant. The pair of equations defining (6) constitutes the "δ-shock Rankine–Hugoniot" conditions for the particular choice, (3).

The first equation of (6) is the standard Rankine–Hugoniot condition and gives the velocity of the δ-shock wave. This velocity verifies the classical entropy condition, (5), and means that all characteristics on both sides of the discontinuity, are in-coming.

The right-hand side of the second equation in (6) is less classical: it is called the "Rankine–Hugoniot deficit". In [8], it is demonstrated that the meaning of amplitude, $e(t)$, of δ-function in v, is the area under the graph $y = v(x_s(t), t)$.

Thus, the system of the Rankine–Hugoniot conditions, (6), determines the trajectory $x = x_s(t)$ of a δ-shock wave and the coefficient $e(t)$ of the singularity.

(b) If $u^l < u^r$, then, the weak solution to the Cauchy problem (2) and (3), is the following one, according to [7]:

$$(u(x,t), v(x,t)) = \begin{cases} (u^l, v^l) & x \leqslant f'(u^l) \times t \\ \left(\frac{x}{2t}, 0\right) & f'(u^l) \times t \leqslant x \leqslant f'(u^r) \times t \\ (u^r, v^r) & x \geqslant f'(u^r) \times t \end{cases} \tag{7}$$

In such a case, the first component, u, of solution (7), is the rarefaction wave while the second component, v, contains the intermediate "vacuum state": $v \equiv 0$.

In this article, we use those theoretical results to construct a HLL Riemann solver that is compatible with the existence of a δ-shock solution in (2).

1.2. Constructing a HLL Riemann solver for weakly hyperbolic systems

Recently, we developed a Hermite Least Square Monotone (HLSM) interpolation technique, in a finite-volume framework. This procedure aims at generating a compact high-order numerical method for systems of hyperbolic conservation laws [9]. For that purpose, we defined as discrete unknowns not only the primitive variable, u, but also its first spatial derivative, $v(\equiv u_x)$. Primitively, those two quantities are defined as solutions of a weakly hyperbolic system, identical to (2) and are evolved in time by using an approximate HLL Riemann solver, simply extended from [10].