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ScienceDirect
Journal of Computational Physics 225 (2007) 1810–1826
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

Tracking discontinuities in hyperbolic conservation laws with spectral accuracy

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Received 28 July 2006; received in revised form 19 February 2007; accepted 19 February 2007
Available online 27 February 2007

Abstract

It is well known that the spectral solutions of conservation laws have the attractive distinguishing property of infinite-order convergence (also called spectral accuracy) when they are smooth (e.g., [C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods for Fluid Dynamics, Springer-Verlag, Heidelberg, 1988; J.P. Boyd, Chebyshev and Fourier Spectral Methods, second ed., Dover, New York, 2001; C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods: Fundamentals in Single Domains, Springer-Verlag, Berlin Heidelberg, 2006]). If a discontinuity or a shock is present in the solution, this advantage is lost. There have been attempts to recover exponential convergence in such cases with rather limited success. The aim of this paper is to propose a discontinuous spectral element method coupled with a level set procedure, which tracks discontinuities in the solution of nonlinear hyperbolic conservation laws with spectral convergence in space. Spectral convergence is demonstrated in the case of the inviscid Burgers equation and the one-dimensional Euler equations.

Keywords: Discontinuous Galerkin; Spectral method; Front tracking; Level set; Ghost fluid method; Hyperbolic systems; Tracking method; Schemes; Fronts

1. Introduction

Numerical methods for treating shocked solutions of conservation laws can be classified into three categories – shock capturing, shock fitting and shock tracking. In a shock-capturing method, the shock is captured automatically by the discretization scheme and the explicit or implicit numerical dissipation (see [4] for an excellent review). In a shock-fitting method, a boundary-fitted co-ordinate system is used, and the shock is treated as a boundary, for which a separate evolution equation is derived and solved (e.g., [5,6]). In a shock-tracking method, the shock is ‘‘captured’’ and identified with a level set function, which is then tracked [7].

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2007.02.016

A shock-capturing method based on spectral discretization exhibits global oscillations in the shocked solutions, known as Gibbs phenomenon [1–3], which usually causes numerical instability, particularly for nonlinear problems. Explicit numerical dissipation is required to obtain stable spectral solutions [8,9]. A stable spectral solution to the quasi-one-dimensional shocked flow was obtained with explicit numerical dissipation represented by a second-order viscous term discretized by central differences [8], and by a spectrally discretized hyperviscosity term [9]. However, spectral accuracy is obtained only away from the shock. Later, post-processing procedures, such as filters [10,11], were employed to smooth out the Gibbs oscillations. These artifacts degrade the spectral accuracy, at least in the vicinity of the shock. As Boyd [12] pointed out, it requires a spatially-adaptive filter whose order varies from small values in the neighborhood of the shock to large values in the smooth regions far away from the shock. To carry out such adaptive filtering, one needs a tool for identifying shocks that is spectrally accurate. There are two other types of post-processing procedures – spectral mollification, and Gegenbauer reconstruction (see [11,13] for critical reviews). The shock-fitting method [14,15] precludes Gibbs phenomenon and provides smooth solutions with spectral accuracy. However, topological difficulty arises in treating complex flow configurations with shocks that are not amenable to being contained in rectangular domains. As the level set formulation [16] has the flexibility to deal with complex topological changes arising in the evolution of curves and surfaces, the objective of the ongoing research has been to couple a level set procedure with the discontinuous spectral element method (DSEM) to deal with discontinuous solutions with spectral accuracy. As a first step towards this end, the effort reported in [17] developed a level set advection algorithm with spectral accuracy. Grooss and Hesthaven [18] propose a spectral Galerkin/level set method for free surface flows, but their "smeared" interface treatment at the discontinuity is only first-order accurate.

In this paper, we present a coupled discontinuous spectral element (DSEM)/level set method that yields uniformly spectral spatial convergence of the solution, including the shock speed and location. Specifically, the computational domain is divided into elements wherein the solution, if it is smooth, is represented by pth-order polynomials (and utilizes appropriate Gauss quadrature techniques). The elements containing a discontinuity (i.e., where the level set function ψ changes sign) are called Godunov elements. Each Godunov element is subdivided into 2^p subelements, and each subelement is treated with a first-order accurate method. So, the "virtual" order of accuracy in a Godunov element is proportional to $(1/2)^p$, which implies spectral accuracy. (Note that in principle, in order to obtain spectral accuracy, we can subdivide a Godunov element into the integer part of a^p subelements for any value of $a > 1$.)

The proposed method may be viewed in two different ways: it may be viewed as a form of (i) adaptive mesh refinement (AMR) [19–22] in which the mesh size h is refined or (ii) p refinement in which the order p of elements is increased. The distinction between the present method and previous methods based on either h refinement or p refinement is that the resulting order of accuracy (as distinguished from the "formal" order of accuracy) is "optimal" in that the order of accuracy of the proposed method is *uniformly* spectral. In other words, the accuracy of the solution in the regular elements as well as the "virtual" order of accuracy in the Godunov elements are both spectral. We remark that for conventional AMR implementations, one must implement complicated interpolation procedures when providing boundary conditions for the refined regions. In the present approach, we exploit the locally high-order accurate representation of the solution in the regular elements to provide boundary conditions for the Godunov elements. We use the piecewise-constant representation in Godunov elements for providing the boundary conditions for the regular elements. We also remark that the present method is distinct from the standard p refinement approaches as it does not retain the high order polynomial representation in the Godunov elements; instead it switches to piecewise constant interpolation in elements containing the zero level set. We emphasize that one must use the level set function to explicitly track the front, since shock capturing (or shock-detection) schemes do not ensure spectral accuracy for shock location.

2. Governing equations

The generic form for a conservation law is

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}+\frac{\partial G(U)}{\partial y}+\frac{\partial H(U)}{\partial z}=0, \tag{1}$$