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Arbitrary high order non-oscillatory finite volume schemes on unstructured meshes for linear hyperbolic systems

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Abstract

In this article we present a non-oscillatory finite volume scheme of arbitrary accuracy in space and time for solving linear hyperbolic systems on unstructured grids in two and three space dimensions using the ADER approach. The key point is a new reconstruction operator that makes use of techniques developed originally in the discontinuous Galerkin finite element framework. First, we use a hierarchical orthogonal basis to perform reconstruction. Second, reconstruction is not done in physical coordinates, but in a reference coordinate system which eliminates scaling effects and thus avoids ill-conditioned reconstruction matrices. In order to achieve non-oscillatory properties, we propose a new WENO reconstruction technique that does not reconstruct point-values but entire polynomials which can easily be evaluated and differentiated at any point. We show that due to the special reconstruction the WENO oscillation indicator can be computed in a mesh-independent manner by a simple quadratic functional. Our WENO scheme does not suffer from the problem of negative weights as previously described in the literature, since the linear weights are not used to increase accuracy. Accuracy is obtained by merely putting a large linear weight on the central stencil. The resulting one-step ADER finite volume scheme obtained in this way performs only one nonlinear WENO reconstruction per element and time step and thus can be implemented very efficiently even for unstructured grids in three space dimensions. We show convergence results obtained with the proposed method up to sixth order in space and time on unstructured triangular and tetrahedral grids in two and three space dimensions, respectively.

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1. Introduction

The goal of our presented work is to construct a numerical method that can solve hyperbolic PDEs with, at least theoretically, arbitrary high order of accuracy in space and time in complex two and three-dimensional domains. Hence, the method should be able to run on unstructured meshes. Therefore, high order finite difference (FD) schemes are not an option due to their requirement of structured grids. Furthermore, the scheme should be able to treat discontinuous solutions without producing spurious oscillations. Finite volume (FV) methods have the advantage over finite difference schemes that they can be extended to high order of accuracy even on unstructured grids using a reconstruction operator. Previous work documented in the literature contains the use of linear reconstruction operators such as least-squares reconstruction [4,30], based on a single stencil. These linear operators, however, generate spurious oscillations in the vicinity of discontinuities. Therefore, nonlinear ENO reconstructions on unstructured grids have been introduced [1,37], as well as WENO reconstructions [23,36,17,27]. However, all the previously cited reconstruction operators have only been used in two space dimensions and the maximum achieved order documented in those publications was four. To our knowledge, no results of three-dimensional ENO or WENO schemes on unstructured tetrahedral grids have been published up to now.

In this article we first present a new linear polynomial reconstruction operator that uses hierarchical orthogonal basis functions in a reference coordinate system to rule out scaling effects and to obtain a very general formulation that facilitates implementation in two and three dimensions and can automatically achieve any desired order of accuracy. The details are given in Section 2. To obtain a non-oscillatory scheme, a nonlinear WENO reconstruction operator is subsequently constructed, based on the linear reconstruction applied to a set of stencils which are then weighted in a nonlinear, solution-dependent way. Due to the use of the special basis functions, the construction and implementation of the resulting WENO scheme can be done very easily even in three space dimensions, see Section 2.3. Once the reconstruction is available, an arbitrary high order accurate finite volume scheme can be constructed using the ADER approach of Toro et al. In this article we focus our considerations on the nonlinear reconstruction operator and fundamental numerics and therefore restrict the applications to linear hyperbolic systems.

In the ADER approach, the numerical flux function is based on the solution of generalized Riemann problems, where the initial data on both sides of the element interfaces are no longer piecewise constant as in the original approach of Godunov [18], but where the initial data is piecewise polynomial, in general separated by a jump at the interface. First ideas of this concept can be traced back to Ben-Artzi and Falcovitz [5], who developed a second-order FV scheme based on the solution of generalized Riemann problems. The idea of arbitrary high order generalized Riemann solvers was first developed by Toro et al. in a finite volume framework for linear equations on Cartesian grids [42,35,34]. They called their approach *ADER*, as abbreviation for "*A*rbitrary high order schemes using *der*ivatives". The extension to nonlinear hyperbolic conservation laws with source terms has then been achieved by Titarev and Toro using a WENO reconstruction technique on Cartesian grids [40,44,41,45]. With the work of Käser and Iske [27] the ADER finite volume approach was for the first time applied on unstructured meshes. They considered nonlinear scalar hyperbolic conservation laws and achieved fourth order of accuracy in space and time using a WENO reconstruction.

The fundamental ideas behind the generalized Riemann problem solvers are a temporal Taylor series expansion of the state at the interface, where then time derivatives are replaced by space derivatives using repeatedly the governing conservation law in differential form, which is the so-called Cauchy–Kovalewski or Lax–Wendroff procedure. However, the problem is that in general neither the state nor the derivatives are defined on the element interfaces where jumps are admitted. The idea is now to solve conventional homogeneous Riemann problems for the state and all space derivatives. This strategy defines the values of the state and the space derivatives on the element interfaces which can then be plugged into the Cauchy–Kovalewski procedure. In the linear case, special simplifications can be applied to increase efficiency. The construction of the scheme, called ADER-FV scheme in this article, is presented in detail in Section 3. It has uniform accuracy in space and time and is stable up to a Courant number of one in one space dimension [16]. Numerical convergence studies are performed up to seventh order of accuracy in space and time on an irregular triangular grid in two dimensions and up to sixth order on a regular tetrahedral grid in three dimensions, see Section 4. The non-oscillatory properties are finally studied on irregular unstructured two and three-dimensional grids in Section 5.