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Computing Arnol′d tongue scenarios

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Abstract

A famous phenomenon in circle-maps and synchronisation problems leads to a two-parameter bifurcation diagram commonly referred to as the Arnol′d tongue scenario. One considers a perturbation of a rigid rotation of a circle, or a system of coupled oscillators. In both cases we have two natural parameters, the coupling strength and a detuning parameter that controls the rotation number/frequency ratio. The typical parameter plane of such systems has Arnol′d tongues with their tips on the decoupling line, opening up into the region where coupling is enabled, and in between these Arnol′d tongues, quasi-periodic arcs. In this paper, we present unified algorithms for computing both Arnol′d tongues and quasi-periodic arcs for both maps and ODEs. The algorithms generalise and improve on the standard methods for computing these objects. We illustrate our methods by numerically investigating the Arnol′d tongue scenario for representative examples, including the well-known Arnol′d circle map family, a periodically forced oscillator caricature, and a system of coupled Van der Pol oscillators.

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1. Introduction

Many interesting problems in science and engineering lead to models involving either periodically forced oscillators or coupled oscillators. Natural parameters to vary in the periodically forced oscillator setting are the forcing amplitude and the forcing period/frequency. In the coupled oscillator setting, coupling strength is a natural parameter, with a typical second parameter, often referred to as a "detuning" parameter, controlling the relative frequencies of the two coupled oscillators. The two settings can be unified by viewing periodically forced oscillators as coupled oscillators, with one-way coupling – the forcing amplitude corresponding to the coupling strength.

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The most prominent phenomenon in these systems is the transition between phase locking (also called entrainment or synchronisation) and quasi-periodicity. Phase locking produces a periodic solution which generically persists as parameters are varied. In contrast, quasi-periodicity is a codimension-one phenomenon which is thus generically destroyed by perturbation. The result is a well-known bifurcation diagram in the two-parameter plane called the "Arnol′d tongue" scenario [1–6,14,17,18,21,26,28,29,37]. It has a countable collection of Arnol′d tongues, emanating from "rational" points on the zero forcing/coupling axis, and opening up into regions where the coupling strength is turned on. Each tongue corresponds to phase locked solutions for which the two frequencies of the oscillators satisfy $\omega_1/\omega_2 = p/p$ for some integers p and q. In between the tongues, emanating from all the "irrational" points on the zero forcing/amplitude axis, are curves of parameters corresponding to quasi-periodic flow on a torus with an irrational frequency ratio ω_1/ω_2. This scenario is generic for weakly coupled oscillators [1,5]. A similar – but not identical – Arnold tongue scenario occurs in the neighbourhood of a Neimark–Sacker curve [6,26,37]. We focus in this paper on continuation from zero forcing amplitude, but arrive at a Neimark–Sacker curve by continuation in the second and third of our three examples in Section 4. Look ahead to examples of these two-parameter bifurcation diagrams in Figs. 5, 9, 13.

There is a variety of ways in which we can model coupled oscillators. The simplest is as a flow in $\mathbb{S} \times \mathbb{S}$. Embedding each oscillator in $\mathbb{R}^{n_i}$, $i = 1,2$, leads to the more general setting of a flow in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. In the decoupled case this flow has an invariant two-torus, which is the product of two limit cycles of the individual oscillators. Assuming these limit cycles are hyperbolic attractors, this two-torus will persist, at least for small coupling strengths. This flow in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is often studied by reduction to a Poincaré return map of $\mathbb{R}^{n_1+n_2-1}$ by sampling the state of the system, for example, as it passes in a specified direction through a well-chosen hyperplane. In the periodically forced oscillator case, this return map can be further reduced to a simple stroboscopic map of the flow in $\mathbb{R}^{n_1}$ at the time period of the uncoupled limit cycle in $\mathbb{R}^{n_2}$. This is possible because the flow in $\mathbb{R}^{n_2}$ is decoupled from the flow in $\mathbb{R}^{n_1}$. The reduction can also be thought of as from a periodic non-autonomous flow in $\mathbb{R}^{n_1}$ to an autonomous map in $\mathbb{R}^{n_1}$. The invariant two-torus in the original flow becomes an invariant circle for either the Poincaré map of $\mathbb{R}^{n_1+n_2-1}$ or the stroboscopic map of $\mathbb{R}^{n_1}$. This allows one further reduction, by restricting attention to the invariant circle, from the maps of $\mathbb{R}^n$ to circle maps. This is the motivation for the Arnol′d sine circle map family which we study in Section 4.1.

Because the invariant circle is not guaranteed to persist globally in the parameter space, we study a more general family in Section 4.2. This family is intended to exhibit generic properties of a Poincaré return map generated by a periodically forced planar oscillator. We call this map the periodically forced oscillator caricature map family. It has been studied previously in [27,28,31,32]. Note that both the Arnol′d circle maps and the caricature maps provide a significant computational shortcut by defining the maps directly, rather than requiring integration of differential equations to define each iterate. Our third family, however, a system of two linearly coupled Van der Pol oscillators, is defined directly from the following system of differential equations:

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = \alpha(y - x),$$
$$\ddot{y} + \varepsilon(y^2 - 1)\dot{y} + (1 + \beta)y = \alpha(x - y),$$

We look briefly at this system now, to preview some of the main results of the paper. Specifically, we compare the computation of certain Arnol′d tongues via traditional methods with the computational algorithms introduced in this paper.

The coupled Van der Pol system has been studied previously in [17,34,36]. We re-investigate it in more depth in Section 4.3. This system is in the coupled oscillator setting introduced above, with coupling strength α and the detuning parameter β. Hence, the (β, α) parameter plane exhibits the Arnol′d tongue scenario and our goal is to compute a preferably large set of these tongues. Since the boundaries of an Arnol′d tongue are loci of saddle-node or fold bifurcations [2,3,17], we used the continuation package AUTO [16] to compute such fold curves for several Arnol′d tongues. A standard method for computing an Arnol′d tongue is to locate a periodic point in the tongue, follow it to a saddle-node bifurcation, and then switch to continue the saddle-node bifurcation curve in α and β as the boundary of the tongue. Fig. 1 shows the first six tongues corresponding to the periods 1, 2, 3, 4 and 5. We found it not only very hard to obtain suitable start data, but we were also unable to continue the curves all the way down to the zero coupling line $\alpha = 0$. This is not due to a limitation