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ScienceDirect
Journal of Computational Physics 219 (2006) 912–929
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

A second order accurate projection method for the incompressible Navier–Stokes equations on non-graded adaptive grids

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Received 13 November 2005; received in revised form 1 July 2006; accepted 4 July 2006
Available online 11 September 2006

Abstract

We present an unconditionally stable second order accurate projection method for the incompressible Navier–Stokes equations on non-graded adaptive Cartesian grids. We employ quadtree and octree data structures as an efficient means to represent the grid. We use the supra-convergent Poisson solver of [C.-H. Min, F. Gibou, H. Ceniceros, A supra-convergent finite difference scheme for the variable coefficient Poisson equation on fully adaptive grids, CAM report 05-29, J. Comput. Phys. (in press)], a second order accurate semi-Lagrangian method to update the momentum equation, an unconditionally stable backward difference scheme to treat the diffusion term and a new method that guarantees the stability of the projection step on highly non-graded grids. We sample all the variables at the grid nodes, producing a scheme that is straightforward to implement. We propose two and three-dimensional examples to demonstrate second order accuracy for the velocity field and the divergence free condition in the L^1 and L^∞ norms.

1. Introduction

The incompressible Navier–Stokes equations describe the motion of fluid flows and are therefore used in countless applications in science and engineering. In non-dimensional form these equations read

$$U_t + (U\cdot\nabla)U + \nabla p = \mu\Delta U + F \quad \text{in } \Omega,$$
$$\nabla\cdot U = 0 \quad \text{in } \Omega,$$
$$U|_{\partial\Omega} = U_b \quad \text{on } \partial\Omega,$$

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2006.07.019

where p is the pressure, F is the sum of the external forces and μ is the viscosity coefficient. Ω represents the domain in which the velocity field U is to be found and $\partial\Omega$ denotes the boundary of the domain, where the velocity field can be prescribed. In this paper, we consider the case where $U \cdot n = 0$ on $\partial\Omega$. These equations lack an evolution equation for pressure, which thus only plays a role in ensuring that the velocity field is divergence free. As a consequence, most numerical methods in the primitive variables are fractional methods, i.e. they first solve the momentum equation ignoring the effects of pressure, and then project the velocity onto the divergence free vector space. Starting with the seminal work of Chorin [8], several projection methods have been introduced, see e.g. the work of Kim and Moin [17], Kan [16], Bell et al. [3] and the references therein. The MAC grid configuration [14] used in finite volume methods, where the pressure is stored at the cells' center and where the velocity components are stored at their respective cells' faces, is often the preferred arrangement. This is mainly due to the fact that it produces methods that offer a straightforward mechanism to enforce *discretely* the incompressibility condition $\nabla \cdot u = 0$. However, other arrangements have been shown to produce high order accurate schemes for the velocity field, without enforcing the incompressibility condition at the discrete level (see e.g. the work of E et al. [10], Almgren et al. [2], the review by Brown et al. [6] and the references therein).

Physical phenomena have differences in length scales and numerical approximations on uniform grids are in such cases extremely inefficient in terms of C.P.U. and memory requirement. This stems from the fact that only a small fraction of the domain needs high grid resolution to correctly approximate the solution, while other parts of the domain can produce accurate solutions on coarser grids (for example in regions where the solution experiences smooth variations). As a consequence adaptive mesh refinement strategies, starting with the work of Berger and Oliger [5] for compressible flows, have been proposed in order to concentrate the computational effort where it is most needed. In the original work of Berger et al. [5,4], a fine Cartesian grid is hierarchically embedded into a coarser grid. Almgren et al. [1] then introduced a projection method for the variable density incompressible Navier–Stokes equations on nested grids. Sussman et al. extended this method to two-phase flows [30]. Within this block structured grid approach, a multigrid approach was used to efficiently solve the Poisson equation. The methods on quadtrees/octrees presented in [21,20,22,24] build one linear system of equations that was solved with standard iterative linear solvers [25].

One of the main difficulties in solving the Navier–Stokes equations on irregular grids is in solving the Poisson equation associated with the incompressibility condition. Rather recently, Popinet [24] introduced a Navier–Stokes solver using an octree data structure. In this work, the discretization of the Poisson equation at one cell's center involves cells that are not necessarily adjacent to it. As a consequence, a non-symmetric linear system of equations was obtained and graded octrees only were considered in order to ease the implementation. In this case the linear system was efficiently solved using a multigrid method. Later, Losasso et al. [21] introduced a symmetric discretization of the Poisson equation in the context of free surface flows. In this case, the discretization at one cell's center only involves adjacent cells, therefore producing a symmetric linear system of equations, which is straightforward to solve with a standard preconditioned conjugate gradient method. Moreover, this method is straightforward to implement and does not require any constraint on the grid. This approach produces first order accurate solutions in the case of a non-graded adaptive mesh and is found to be second order accurate in the case of a graded mesh. In this case, the pressure fluxes defined at the faces are the same for a large cell and its adjacent smaller cells. Using ideas introduced in [19], Losasso et al. then extended this method to second order accuracy. In [22], Min et al. introduced a second order accurate method to solve the Poisson equation on non-graded adaptive grids as well. A hallmark of this approach is that the solution's gradients are found to second order accuracy as well. In this case, the linear system is non-symmetric but is proven to be diagonally dominant. In this paper, we propose a second order accurate finite difference Navier–Stokes solver on non-graded adaptive grids, making use of the Poisson solver introduced in [22].

2. Spatial discretization

The physical domain in two (resp. three) spatial dimensions is discretized into squares (resp. cubes), and we use a standard quadtree (resp. octree) data structure to represent this partitioning. For example, consider the case depicted in Fig. 1 in the case of two spatial dimensions: The root of the tree is associated with the entire domain that is then split into four cells of equal sizes, called the children of the root. The discretization