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Journal of Computational Physics 217 (2006) 37–62
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

"Natural norm" a posteriori error estimators for reduced basis approximations

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Received 15 August 2005; received in revised form 27 January 2006; accepted 9 February 2006
Available online 19 April 2006

Abstract

We present a technique for the *rapid* and *reliable* prediction of linear-functional outputs of coercive and non-coercive linear elliptic partial differential equations with affine parameter dependence. The essential components are: (i) rapidly convergent global reduced basis approximations – (Galerkin) projection onto a space W_N spanned by solutions of the governing partial differential equation at N judiciously selected points in parameter space; (ii) a posteriori error estimation – relaxations of the error-residual equation that provide inexpensive yet sharp bounds for the error in the outputs of interest; and (iii) offline/online computational procedures – methods which decouple the generation and projection stages of the approximation process. The operation count for the online stage – in which, given a new parameter value, we calculate the output of interest and associated error bound – depends only on N (typically very small) and the parametric complexity of the problem.

In this paper we propose a new "natural norm" formulation for our reduced basis error estimation framework that: (a) greatly simplifies and improves our inf–sup lower bound construction (offline) and evaluation (online) – a critical ingredient of our a posteriori error estimators; and (b) much better controls – significantly sharpens – our output error bounds, in particular (through deflation) for parameter values corresponding to nearly singular solution behavior. We apply the method to two illustrative problems: a coercive Laplacian heat conduction problem – which becomes singular as the heat transfer coefficient tends to zero; and a non-coercive Helmholtz acoustics problem – which becomes singular as we approach resonance. In both cases, we observe very economical and sharp construction of the requisite natural-norm inf–sup lower bound; rapid convergence of the reduced basis approximation; reasonable effectivities (even for near-singular behavior) for our deflated output error estimators; and significant – several order of magnitude – (online) computational savings relative to standard finite element procedures.

Keywords: Parametrized partial differential equations; Reduced basis methods; Galerkin approximation; Inf–sup constant; Output bounds; A posteriori error estimation; Adjoint methods; Deflation

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2006.02.012

1. Introduction

1.1. Reduced basis approach

Engineering analysis requires the prediction of an (or more realistically, several) "output of interest" $s^e \in \mathbb{R}$ – related to energies or forces, stresses or strains, flowrates or pressure drops, temperatures or fluxes – as a function of an "input" parameter P-vector $\mu \in \mathcal{D} \subset \mathbb{R}^P$ – related to geometry, physical properties, boundary conditions, or loads. These outputs $s^e(\mu)$ are often functionals of a field variable $u^e(\mu)$,

$$s^e(\mu) = \ell(u^e(\mu)), \tag{1}$$

where $u^e(\mu) \in X^e$ – say displacement, velocity, or temperature – satisfies in weak form the μ-parametrized (elliptic linear) partial differential equation

$$a(u^e(\mu), v; \mu) = f(v) \quad \forall v \in X^e. \tag{2}$$

Here X^e is the appropriate function space, and a (respectively ℓ, f) are continuous bilinear (respectively, linear) forms.

In general, we cannot find the exact (our superscript "e" above) solution, and hence we replace $s^e(\mu)$, $u^e(\mu)$ with a Galerkin finite element approximation, $s^{\mathcal{N}}(\mu), u^{\mathcal{N}}(\mu)$: given $\mu \in \mathcal{D}$,

$$s^{\mathcal{N}}(\mu) = \ell(u^{\mathcal{N}}(\mu)), \tag{3}$$

where $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}}$ satisfies

$$a(u^{\mathcal{N}}(\mu), v; \mu) = f(v) \quad \forall v \in X^{\mathcal{N}}. \tag{4}$$

Here $X^{\mathcal{N}} \subset X^e$ is a standard finite element approximation subspace of dimension $\mathcal{N}$. Unfortunately, to achieve the desired accuracy, $\mathcal{N}$ must typically be chosen very large; as a result, the evaluation $\mu \rightarrow s^{\mathcal{N}}(\mu)$ is simply too costly in the many-query and real-time contexts often of interest in engineering. Low-order models – we consider here reduced basis approximations – are thus increasingly popular in the engineering analysis, parameter estimation, design optimization, and control contexts.

In the reduced basis approach [1–7], we approximate $s^{\mathcal{N}}(\mu), u^{\mathcal{N}}(\mu)$ – for some fixed sufficiently large "truth" $\mathcal{N} = \mathcal{N}_t$ – with $s_N(\mu)$, $u_N(\mu)$: given $\mu \in \mathcal{D}$,

$$s_N(\mu) = \ell(u_N(\mu)), \tag{5}$$

where $u_N(\mu) \in W_N$ satisfies[1]

$$a(u_N(\mu), v; \mu) = f(v) \quad \forall v \in W_N. \tag{6}$$

Here W_N is a problem-specific space of dimension $N \ll \mathcal{N}_t$ that focuses on the (typically very smooth) parametric manifold of interest – $\{u^{\mathcal{N}_t}(\mu) | \mu \in \mathcal{D}\}$ – and thus enjoys very rapid convergence $u_N(\mu) \rightarrow u^{\mathcal{N}_t}(\mu)$ and hence $s_N(\mu) \rightarrow s^{\mathcal{N}_t}(\mu)$ as N increases [3,8]. This dramatic *dimension reduction*, in conjunction with *offline/online computational procedures* [6,7,9,10], yields very large savings in the many-query and real-time contexts: the online complexity depends only on the size of the reduced basis space, N, which is typically orders of magnitude smaller than the dimension of the finite element space, $\mathcal{N}_t$.

Our own effort is dedicated to the development of a posteriori error estimators for reduced basis approximations [6,7,11,12]: inexpensive – complexity *independent* of $\mathcal{N}_t$ – and sharp error bounds $\Delta_N^s(\mu)$ such that

$$|s^{\mathcal{N}_t}(\mu) - s_N(\mu)| \leqslant \Delta_N^s(\mu) \quad \forall \mu \in \mathcal{D}.$$

Absent such rigorous error bounds we cannot efficiently determine if N is too small – and our reduced basis approximation unacceptably inaccurate – or if N is too large – and our reduced basis approximation unnecessarily expensive. (Furthermore, in the nonlinear context, error bounds are crucial in establishing the very *existence* of a "truth" solution $u^{\mathcal{N}_t}(\mu)$ [13–15].) We cannot determine in "real-time" if critical design conditions and constraints are satisfied – for example, does approximate feasibility $s_N(\mu) \leqslant C$ imply "true" feasibility $s^{\mathcal{N}_t}(\mu) \leqslant C$? And, in fact, we can not even construct an efficient and well-conditioned reduced basis approximation space W_N [12,16].

[1] For simplicity in this Introduction, we consider a purely primal approach; we shall subsequently pursue a primal–dual formulation.