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High-order triangle-based discontinuous Galerkin methods for hyperbolic equations on a rotating sphere

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Abstract

High-order triangle-based discontinuous Galerkin (DG) methods for hyperbolic equations on a rotating sphere are presented. The DG method can be characterized as the fusion of finite elements with finite volumes. This DG formulation uses high-order Lagrange polynomials on the triangle using nodal sets up to 15th order. The finite element-type area integrals are evaluated using order $2N$ Gauss cubature rules. This leads to a full mass matrix which, unlike for continuous Galerkin (CG) methods such as the spectral element (SE) method presented in Giraldo and Warburton [A nodal triangle-based spectral element method for the shallow water equations on the sphere, J. Comput. Phys. 207 (2005) 129–150], is small, local and efficient to invert. Two types of finite volume-type flux integrals are studied: a set based on Gauss–Lobatto quadrature points (order $2N-1$) and a set based on Gauss quadrature points (order $2N$). Furthermore, we explore conservation and advection forms as well as strong and weak forms. Seven test cases are used to compare the different methods including some with scale contractions and shock waves. All three strong forms performed extremely well with the strong conservation form with $2N$ integration being the most accurate of the four DG methods studied. The strong advection form with $2N$ integration performed extremely well even for flows with shock waves. The strong conservation form with $2N-1$ integration yielded results almost as good as those with $2N$ while being less expensive. All the DG methods performed better than the SE method for almost all the test cases, especially for those with strong discontinuities. Finally, the DG methods required less computing time than the SE method due to the local nature of the mass matrix.
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1. Introduction

On spherical domains, the most natural solution strategy is to use spherical harmonics (spectral transform methods) on a Gaussian grid where the longitude and latitude are the spherical coordinates. However, choosing spherical harmonics eliminates any possibility of exploiting adaptive solution strategies. Furthermore, for

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solving relevant problems the numerical model must be run in a distributed-memory mode (such as with the message-passing interface). It is well known that the cost of spherical harmonics is $O(N_{\mathrm{lat}}^3)$ where N_{lat} denotes the number of grid points in the latitudinal direction (south to north pole).

On the other hand, local methods (e.g., finite differences, elements, and volumes) cost on the order of $O(N_{\mathrm{p}}^2)$, where N_{p} denotes the number of total grid points. If either adaptivity or unstructured grids are to be used then this now only leaves finite elements (FE) and finite volumes (FV) as the only two viable options. Typically, a choice has had to be made between high order accuracy and local conservation.

If high order accuracy (beyond 2nd order) is selected as the main criterion then the FE method must be the method selected. High order FE methods are typically referred to as spectral elements (SE) and we shall use these two terms interchangeably throughout this manuscript. FE/SE methods have shown to be quite capable of producing very accurate solutions for flows on rotating spheres (see [16]) provided that the solutions are smooth. However, if the solutions are non-smooth then FE/SE methods do not perform as well. We showed this in [13] in the context of unstructured quadrilateral elements and we show this in Section 5 for unstructured triangular elements.

However, if local conservation is the main criterion then FV methods must be chosen. FV methods have been shown to be quite effective in handling discontinuous flows on rotating spheres (see [24]). However, FV methods are at most second order accurate on unstructured triangular grids (see [10]); higher order reconstructions are only readily available for Cartesian (structured) grids and only using quadrilaterals.

Thus if both high order accuracy and local conservation on unstructured triangular grids are sought then the natural choice is the discontinuous Galerkin (DG) method. In essence, the DG method extracts the best features of FE and FV methods and fuses them into a powerful method capable of delivering high order accuracy in conjunction with local conservation. In [13], we introduced the first DG formulation for flows on a rotating sphere using much of the same machinery originally developed for SE methods (see [12]); the main difference being that we replaced the C^0 continuity condition of SE methods with a discontinuity at the element interfaces resolved via jump conditions in a similar vein to that of penalty methods (see [4]). Because the DG method shares much in common with FV methods then much of the same machinery developed for FV methods such as Riemann solvers, total variation diminishing (TVD) schemes, and nonlinear flux limiters can be applied to DG methods which then renders the solutions not only high-order accurate but also monotonicity preserving as well.

In addition to offering local conservation, high order accuracy, monotonicity, and adaptivity, the DG method also offers efficiency and natural *parallelization* especially for unstructured triangular grids. To clarify this point, in Fig. 1 we show the discrete stencil required by both the DG and SE methods. The DG stencil is

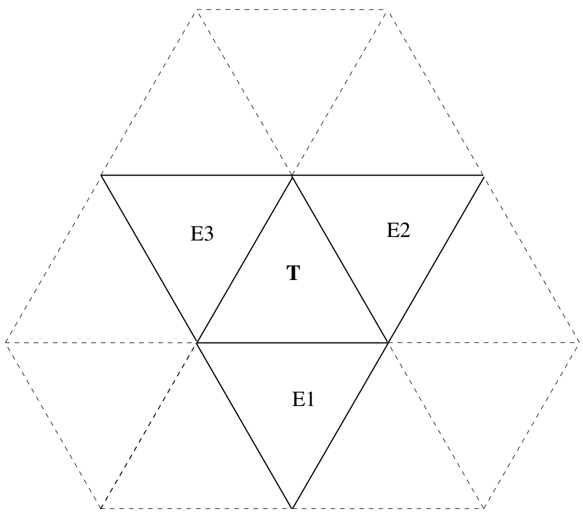

Fig. 1. The discrete stencil of the triangle, T, for the DG method (solid triangles) and the SE method (solid and dashed triangles).