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A proof that the discrete singular convolution (DSC)/Lagrange-distributed approximating function (LDAF) method is inferior to high order finite differences

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Abstract

Finite differences approximate the mth derivative of a function $u(x)$ by a series $\sum_{j=-N}^{N} d_j^{(m)} u(x_j)$, where the x_j are the grid points. The closely-related discrete singular convolution (DSC) and Lagrange-distributed approximating function (LDAF) methods, treated here as a single algorithm, approximate derivatives in the same way as finite differences but with different numerical weights that depend upon a free parameter a. By means of Fourier analysis and error theorems, we show that the DSC is worse than the standard finite differences in differentiating $\exp(ikx)$ for all k when $a \geqslant a_{\mathrm{FD}}$ where $a_{\mathrm{FD}} \equiv 1/\sqrt{N+1}$ with N as the stencil width is the value of the DSC parameter that makes its weights most closely resemble those of finite differences. For $a < a_{\mathrm{FD}}$, the DSC errors are *less* than finite differences for k near the aliasing limit, but much, much worse for smaller k. Except for the very unusual case of low-pass filtered functions, that is, functions with negligible amplitude in small wavenumbers k, the DSC/LDAF is less accurate than finite differences for all stencil widths N. So-called "spectrally-weighted" or "frequency-optimized" differences are superior for this special case. Consequently, DSC/LDAF methods are *never* the best way to approximate derivatives on a stencil of a given width.

Keywords: Pseudospectral; Lagrange-distributed approximating functional; Discrete singular convolution; High order finite difference; Nonstandard finite differences; Spectral differences

1. Introduction

A pseudospectral method is a finite difference method in which the stencil is the entire grid [10,15]. Boyd [5] showed that one could derive both standard finite differences and also a great variety of nonstandard difference schemes by applying "sum acceleration" or "summability" methods to the pseudospectral difference

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sums. Sum acceleration methods apply running averages or similar artifices to the slowly-converging pseudospectral series so that these can be truncated to stencils of finite width without a drastic loss of accuracy.

The linear distributed approximating functional (LDAF) method was invented in 1991 by Hoffman et al. [20]. By 1997, the original concept had been refined to use a basis in which each element is the product of a Lagrangian interpolating cardinal function (as in classic finite differences) but multiplied by a Gaussian tapering function. Wei replaced the polynomial by the sinc function and dubbed the result the discrete singular convolution (DSC) method [3,13,41,43]. It is impossible to review the work of this very prolific author in detail, but of the more than one hundred articles listed at http://www.math.msu.edu/wei/, most use the DSC or LDAF schemes. Since the sinc function is closely related to polynomial interpolation as lucidly explained in [14,15], the Gaussian-weighted LDAF and DSC are so closely related that it is sufficient for our purposes to treat them as one.

Although the LDAF/DSC literature justifies the method through a rather elaborate machinery of smoothed Delta functions, Schwarz distribution theory and so on, the end result is that derivatives are approximated by formulas of the same form as classical finite differences [4] except that the numerical weights of the grid point values are different. The LDAF/DSC is also a special case of Boyd's earlier theory of sum-accelerated pseudospectral methods: special in that the weighting function is a Gaussian. Is this a good weight?

In the remainder of this article, we answer a resounding: No! In Sections 2 and 3, we review two essential background technologies: Fourier analysis of derivative approximations and sum-accelerated sinc pseudospectral methods. In Section 4, we show that by using this formalism, one can derive standard finite differences by a non-standard route and prove a rigorous theorem for the error in differentiating $\exp(\mathrm{i}kx)$. In the following section, we derive a similar theorem for the error in the DSC method.

2. Fourier analysis

Fourier analysis has been widely used to analyze difference formulas ever since this was popularized by von Neuman. The reason is that the Fourier basis function, $\exp(\mathrm{i}kx)$, is an eigenfunction of both the differentiation operator and also of all possible difference formulas. This implies that the accuracy of difference formulas can be assessed – and improved – merely by comparing the eigenvalues.

Theorem 2.1. *Let h denote the spacing of a uniform grid:*

$$x_j \equiv jh. \tag{1}$$

Define a grid-scaled differentiation operator:

$$D_m \equiv \mathrm{i}^{-m} h^m \frac{\mathrm{d}^m}{\mathrm{d}x^m}, \tag{2}$$

where $\mathrm{i} = \sqrt{-1}$. Introduce the centered approximation

$$D_m^{\mathrm{app}} u(x) \equiv \sum_{j=-N}^{N} d_j^{(m)} u(x + jh), \tag{3}$$

where "centered" implies

$$d_j^m = (-1)^m d_{-j}^m. \tag{4}$$

The set of $(2N + 1)$ points $x_j \equiv jh$, $j = -N, \ldots, N$, is said to be the "stencil" of the difference formula. Define the scaled wavenumber

$$K \equiv kh. \tag{5}$$

Then the differentiation operator has the exact eigenrelation

$$D_m \exp(\mathrm{i}kx) \equiv K^m \exp(\mathrm{i}Kx/h) \tag{6}$$

and the difference operator has the exact eigenrelation