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High order well-balanced finite volume WENO schemes and discontinuous Galerkin methods for a class of hyperbolic systems with source terms

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Abstract

Hyperbolic balance laws have steady state solutions in which the flux gradients are nonzero but are exactly balanced by the source term. In our earlier work [J. Comput. Phys. 208 (2005) 206–227; J. Sci. Comput., accepted], we designed a well-balanced finite difference weighted essentially non-oscillatory (WENO) scheme, which at the same time maintains genuine high order accuracy for general solutions, to a class of hyperbolic systems with separable source terms including the shallow water equations, the elastic wave equation, the hyperbolic model for a chemosensitive movement, the nozzle flow and a two phase flow model. In this paper, we generalize high order finite volume WENO schemes and Runge–Kutta discontinuous Galerkin (RKDG) finite element methods to the same class of hyperbolic systems to maintain a well-balanced property. Finite volume and discontinuous Galerkin finite element schemes are more flexible than finite difference schemes to treat complicated geometry and adaptivity. However, because of a different computational framework, the maintenance of the well-balanced property requires different technical approaches. After the description of our well-balanced high order finite volume WENO and RKDG schemes, we perform extensive one and two dimensional simulations to verify the properties of these schemes such as the exact preservation of the balance laws for certain steady state solutions, the non-oscillatory property for general solutions with discontinuities, and the genuine high order accuracy in smooth regions.

Keywords: Hyperbolic balance laws; WENO scheme; Discontinuous Galerkin method; High order accuracy; Source term; Conservation laws; Shallow water equation; Elastic wave equation; Chemosensitive movement; Nozzle flow; Two phase flow

1. Introduction

In this paper, we are interested in designing high order weighted essentially non-oscillatory (WENO) finite volume schemes and Runge–Kutta discontinuous Galerkin (RKDG) finite element methods for solving hyperbolic systems of conservation laws with source terms (also called balance laws)

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$$u_t + f_1(u,x,y)_x + f_2(u,x,y)_y = g(u,x,y) \tag{1.1}$$

or in the one dimensional case

$$u_t + f(u,x)_x = g(u,x), \tag{1.2}$$

where u is the solution vector, $f_1(u,x,y)$ and $f_2(u,x,y)$ (or $f(u,x)$) are the fluxes and $g(u,x,y)$ (or $g(u,x)$) is the source term.

These balance laws often admit steady state solutions in which the source term is exactly balanced by the flux gradients. Such cases, along with their perturbations, are very difficult to capture numerically. A straightforward treatment of the source terms will fail to preserve this balance. The objective of well-balanced schemes is to preserve exactly some of these steady state solutions. This objective should be achieved without sacrificing the high order accuracy and non-oscillatory properties of the scheme when applied to general, non-steady state solutions.

A typical example considered extensively in the literature for balance laws is the shallow water equation with a non-flat bottom topology. Research on numerical methods for the solution of the shallow water system has attracted significant attention in the past two decades. An early, important result in computing such solutions was given by Bermudez and Vazquez [3]. They proposed the idea of the "exact C-property", which means that the scheme is "exact" when applied to the stationary case $h + b = \text{constant}$ and $hu = 0$, where h, b and u are the water height, the given bottom topography, and the velocity of the fluid, respectively, see (6.1) in Section 6.1. A good scheme for the shallow water system should satisfy this property. Also, Bermudez and Vazquez introduced in [3] the first order Q-scheme and the idea of source term upwinding. After this pioneering work, many other schemes for the shallow water equations with such well-balanced property have been developed. LeVeque [21] introduced a quasi-steady wave propagation algorithm. A Riemann problem is introduced in the center of each grid cell such that the flux difference exactly cancels the source term. Zhou et al. [38] used the surface gradient method for the treatment of the source terms. They used $h + b$ for the reconstruction instead of using h. For more related work, see also [13,14,17,19,20,24,26,33,34,37]. In particular, the authors of [33,34] presented well-balanced ENO and WENO schemes for the shallow water equations and other equations.

Our development of well-balanced WENO finite volume schemes and discontinuous Galerkin methods is based on our recent work [35,36]. In [35], we developed a well-balanced high order finite difference WENO scheme for solving the shallow water equation, which is non-oscillatory, well balanced (satisfying the exact C-property) for still water, and genuinely high order in smooth regions. Different from [33,34], a key ingredient of the technique used in [35] is a special decomposition of the source term, allowing a discretization to the source term to be both high order accurate for general solutions and exactly well balanced with the flux gradient for still water. Extensive one and two dimensional numerical experiments were provided in [35] to demonstrate the good behavior of this scheme. In [36], we extended this idea of decomposition of source terms to a general class of balance laws with separable source terms, allowing the design of well-balanced high order finite difference WENO scheme for all balance laws falling into this category. This class is quite general, including, besides the shallow water equations, the elastic wave equation, the hyperbolic model for a chemosensitive movement, the nozzle flow and a two phase flow model.

In this paper, we consider finite volume WENO schemes first introduced by Liu et al. [22], see also [16,27], and Runge–Kutta discontinuous Galerkin (RKDG) finite element methods that were originally developed by Cockburn and Shu [8], see also [7,9]. We will generalize these schemes to obtain high order well-balanced schemes. The crucial difference between the finite volume and the finite difference WENO schemes is that the WENO reconstruction procedure for a finite volume scheme applies to the solution and not to the flux function values. As a consequence, finite volume schemes are more suitable for computations in complex geometry and for using adaptive meshes, however the maintenance of the well-balanced property requires different technical approaches. The RKDG methods can be considered as a generalization of finite volume schemes, even though they do not require a reconstruction and evolve the complete polynomial in each cell forward in time. The RKDG methods are therefore easier to use for multi-dimensional problems in complex geometry, than the finite volume schemes, as the complicated reconstruction procedure can be avoided. Even