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Simultaneous space–time adaptive wavelet solution of nonlinear parabolic differential equations

Jahrul M. Alam [a], Nicholas K.-R. Kevlahan [a,*], Oleg V. Vasilyev [b]

[a] Department of Mathematics & Statistics, McMaster University, 1280 Main St W, Hamilton, ON, Canada L8S 4K1

[b] Department of Mechanical Engineering, University of Colorado, 427 UCB, Boulder, CO 80309, USA

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Abstract

Dynamically adaptive numerical methods have been developed to efficiently solve differential equations whose solutions are intermittent in both space and time. These methods combine an adjustable time step with a spatial grid that adapts to spatial intermittency at a fixed time. The same time step is used for all spatial locations and all scales: this approach clearly does not fully exploit space–time intermittency. We propose an adaptive wavelet collocation method for solving highly intermittent problems (e.g. turbulence) on a simultaneous space–time computational domain which naturally adapts both the space and time resolution to match the solution. Besides generating a near optimal grid for the full space–time solution, this approach also allows the global time integration error to be controlled. The efficiency and accuracy of the method is demonstrated by applying it to several highly intermittent $(1D+t)$-dimensional and $(2D+t)$-dimensional test problems. In particular, we found that the space–time method uses roughly 18 times fewer space–time grid points and is roughly 4 times faster than a dynamically adaptive explicit time marching method, while achieving similar global accuracy.

Keywords: Wavelets; Lifting scheme; Second generation wavelets; Partial differential equations; Elliptic problem; Adaptive grid; Numerical method; Multi-level method; Multi-grid method

1. Introduction

Mathematical modeling of problems in science and engineering (e.g. turbulence, reactive or non-reactive flows [1]) typically involves solving nonlinear partial differential equations (PDEs). A wide range of spatial and temporal scales must often be resolved in order to properly solve these equations [2]. However, in many situations the small spatial scales are highly localized, and thus efficient solution of the problem requires a locally adapted grid. A uniformly fine grid is clearly inefficient for such problems. Turbulence is a well-known example of a problem with high intermittency [3,4]. In high Reynolds number turbulence the number of degrees of freedom scales like the cube of Reynolds number, Re^3, in a uniform mesh that resolves the smallest

* Corresponding author.
E-mail addresses: alamj@math.mcmaster.ca (J.M. Alam), kevlahan@mcmaster.ca (N.K.-R. Kevlahan), Oleg.Vasilyev@Colorado.EDU (O.V. Vasilyev).

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active structures in space and time [5]. Since turbulence usually occurs at high Reynolds number (e.g. $Re \sim 10^6$ for a typical aeronautical flow), it is clear that any successful direct numerical simulation (DNS) of turbulence must take advantage of the flow's high intermittency [6,7]. Because of this intermittency, we expect that the minimum number of computational elements required is actually much smaller than Re^3.

Recently there has been increasing interest in developing adaptive [8–15] numerical methods for solving elliptic [16–20] and time-dependent [21–35] partial differential equations. Existing adaptive numerical methods fall into two classes: *error indicator* based (where the grid is refined to resolve gradients of a physically relevant quantity), and *error control* based (where the error is estimated and the grid is refined to ensure this error is less than a prescribed tolerance). The error-indicating strategy does not control the error directly, but instead controls the mesh coarsening and refinement. The error-estimating strategy minimizes the error as measured in an appropriate norm, which leads to an optimal mesh size distribution.

Wavelets have proved to be an efficient tool in developing adaptive numerical methods which control the global (usually L^2) approximation error [17,18,21,24,26,35]. The goal of the collocation-based nonlinear wavelet approximation is to obtain the best approximation of a function on a near optimal grid. The collocation approximation has a one-to-one correspondence between the wavelet expansion coefficients and grid points. Thus, nonlinear filtering of wavelet coefficients automatically refines the computational grid. Since functions and operators can be computed with a given accuracy, adaptive wavelet method provides global error control for the adaptive solution of differential equations.

Liandrat and Tchamitchian [21] proposed the first wavelet-based adaptive method for partial differential equations. Until the work of Sweldens [36], the research effort was focused on compressing both the differential operators and the solution using Galerkin projection. The early work found in [37–39] demonstrated the use of wavelets to find the numerical solution of PDEs with periodic boundary conditions. Galerkin-based wavelet methods for linear elliptic problems were studied in [16,40–42]. Schneider [43] used reliable and efficient a posteriori error estimates for adaptive multi-scale wavelet-Galerkin schemes for linear elliptic PDEs. The error achieved by adaptive wavelet schemes [16,17,19] is proportional to the smallest error realized by the wavelet approximation, i.e. these schemes are asymptotically optimal for elliptic problems [20]. In addition, adaptive wavelet methods are fast (at least for large problems) since the computational complexity scales like the number of wavelets retained in the approximation, $O(\mathcal{N})$.

Adaptive wavelet schemes have also been used for solving time dependent partial differential equations [21,22]. A more detailed derivation of fast and adaptive algorithms, projection of the solution and spatial derivatives on wavelet space, relationship between sparseness of the discretized system and the vanishing moment property of wavelets was developed by Beylkin and Keiser [29]. Debussche et al. [44] developed a multi-level Fourier–Galerkin method for homogeneous turbulence. The main difficulties of a Galerkin-based wavelet method are the efficient computation of nonlinear operators, and the implementation of general boundary conditions. These difficulties led to the development of collocation-based adaptive wavelet methods, e.g. [25,26,28]. Following the second-generation multi-resolution approximation of Sweldens [45], a multi-level adaptive wavelet collocation method was developed by Vasilyev and Bowman [34], which was applied to a wide variety of initial value problems problems [35]. The adaptive wavelet collocation method (AWCM) has since been used to construct a multi-level adaptive elliptic solver [46], two- and three-dimensional simulation of fluid–structure interaction [47–49], and a wavelet-based alternative to large eddy simulation [50].

To the best of our knowledge, all existing wavelet methods for time-dependent problems adapt the spatial grid dynamically in the region of intermittency. This means that mesh refinement or coarsening is automatic if the solution develops strong gradients, or if these gradients diffuse. If the solution is intermittent in both space and time, one adapts the spatial mesh to the solution at a fixed time and uses an adjustable time step to control the local error in time [22]. This approach enforces the same time step for all spatial locations, which is clearly not optimal for problems which are simultaneously intermittent in both space and time.

Following the classical time marching technique, an adaptive wavelet method discretizes the PDE to produce a system of ODEs with wavelet coefficients as time-dependent unknowns (in the Galerkin formulation), or nodal approximations as time-dependent unknowns in the collocation approach. This method adapts the spatial grid as shocks or any localized structures develop or move in a time-dependent solution. The dynamically adjusted time stepping procedure determines the maximum allowable time step for the spatially adapted grid, or for all the wavelet modes, at any instant. However, although the spatial error is controlled by the adaptive