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# A discrete rational adjustment process of link flows in traffic networks

Ren-Yong Guo [a], Hai Yang [b,c,*], Hai-Jun Huang [c]

[a] *College of Computer Science, Inner Mongolia University, Hohhot 010021, PR China*
[b] *Department of Civil and Environmental Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, PR China*
[c] *School of Economics and Management, Beihang University, Beijing 100191, PR China*

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ABSTRACT

A discrete dynamical system model, called a discrete rational adjustment process (DRAP), is built upon link-based variables to characterize the process of achieving equilibrium from a non-equilibrium state in traffic networks. The equilibrium state can be in either a deterministic or a stochastic user equilibrium state. The DRAP is formulated in a general framework with either fixed or elastic demand. Several mathematical properties of the DRAP are presented, including the invariance of its evolutionary trajectory, the equivalence between its stationary state and user equilibrium, the uniqueness of its stationary state, and its convergence. Proper toll schemes can make the DRAP evolve towards a system optimum state. Numerical experiments are carried out to show application of the DRAP and its properties and the effectiveness of toll schemes.

## 1. Introduction

Traffic assignment analysis, based on the notion of user equilibrium (UE), has been a traditional subject of a stream line of research in the field of transportation. Its fundamental aim is to find flow patterns in a traffic network, given the network topology, origin–destination (OD) trip rates, link performance functions, and the route choice criteria of users. The concept of UE was first described by Wardrop (1952) and formulated as a mathematical programming problem by Beckmann et al. (1956). Subsequently, Smith (1979) and Dafermos (1980) formulated this concept as a variational inequality problem with general asymmetric link costs. This class of UE models has been seen to be versatile, with extensions able to represent the travelers' perceptual variations, mixed vehicle types, and some mixed mode operations. Due to its inability to deal with many of the dynamic aspects of traffic systems, the class of UE models has been extended to represent within-day dynamics.

Subsequently, more fundamental questions were raised about the behavioral basis of the equilibrium paradigm, particularly in the sense of its usefulness in representing travelers' response to exogenous information (Mahmassani and Jayakrishnan, 1991). In fact, travelers adjust their routes from day to day and the resultant link or route flows evolve over time. Moreover, with the development of Advanced Traveler Information Systems (ATIS), travelers can more frequently update their routes with the assistance of traffic information provided by ATIS. This leads to growing interests in studying the day-to-day dynamics with traffic congestion, whereby the evolution of travel choices over days. Such studies can help us well understand the adjustment mechanism of traffic flows in urban traffic networks and develop better user equilibrium models.

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[*] Corresponding author at: Department of Civil and Environmental Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, PR China. Tel.: +852 2358 7178; fax: +852 2358 1534.

*E-mail address:* cehyang@ust.hk (H. Yang).

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So far, the day-to-day dynamics of traffic flows has been formulated as either continuous time processes (e.g., Smith, 1984; Friesz et al., 1994; Zhang and Nagurney, 1996) or discrete time systems (e.g., Horowitz, 1984; Cantarella and Cascetta, 1995; Watling, 1999). In these continuous time processes, adjustment to traffic flows is made in continuous time. However, in reality repeated travels are subject to activity constraints and traffic flows are not continuously adjustable over days. Therefore, a more appropriate representation would seem to be a discrete time system, with each discrete time epoch representing a lag over which travels may be repeated, e.g. days or weekdays (Watling and Hazelton, 2003; Parry and Hazelton, 2013). In addition, the convergences of discrete time and continuous time processes are characterizing rather different properties (Watling, 1999). A process that is convergent in continuous time may not be convergent with respect to any discretization of the process. Convergence in discrete time is not implied either by convergence of some continuous time approximation.

It should be mentioned here that two classes of dynamical system models have been proposed to formulate the evolution of route and link flows from day to day in traffic networks, respectively. Both classes of models share the common property that their stationary states correspond to the UE state, namely, when the traffic flows remain unchanged from day to day, the traffic flows must be in the UE state.

As summarized by Yang and Zhang (2009), five major categories of dynamical systems have been built upon route flow variables to formulate the adjustment behavior of travelers' route choice from day to day. They are the simplex gravity flow dynamics (e.g., Smith, 1983), the proportional-switch adjustment process (e.g., Smith, 1984; Smith and Wisten, 1995; Huang and Lam, 2002; Peeta and Yang, 2003), the network tatonnement process (e.g., Friesz et al., 1994; Jin, 2007; Guo and Huang, 2009), the projected dynamical system (e.g., Zhang and Nagurney, 1996; Nagurney and Zhang, 1997), and the evolutionary traffic dynamics (e.g., Sandholm, 2001; Yang, 2005).

From a practical perspective, the number of links is generally less than that of routes in a traffic network. It is easier to monitor and analyze the link flow dynamics than the route flow dynamics. Current ATIS can provide travelers with the traffic dynamics on links or the static information on routes. However, it is difficult for ATIS to provide reliable route-based dynamic information, as the number of routes is usually much larger. Therefore, it is essential to develop dynamical system models formulating the evolution of link flows from day to day in traffic networks. In this respect, He et al. (2010) proposed a day-to-day traffic assignment model that directly deals with link flow variables and its stationary state is shown to correspond to the deterministic user equilibrium (DUE) state. Han and Du (2012) further performed an analysis of the model by He et al. (2010) and established several properties, including the invariance set and the stability. He and Liu (2012) proposed a prediction–correction model to describe the process realizing traffic equilibration in link variables. A predicted flow pattern was constructed inside the model to accommodate the imperfect perception of congestion that is gradually corrected by actual travel experiences. Guo et al. (2013) proposed a general link-based dynamical system model. The model includes, as special cases, the adjustment process of link flows examined by He et al. (2010) and Han and Du (2012) and the link flow splitting mechanism proposed by Smith and Mounce (2011).

As a unification of and advance over the various types of route-based flow dynamical systems, Zhang et al. (2001) and Yang and Zhang (2009) proposed a type of travelers' day-to-day route choice adjustment behaviors, termed as rational behavior adjustment process (RBAP). Under the RBAP, it is assumed that the mechanism of travelers' behavior in their route choice to avoid higher travel cost paths will push the system in a direction to reduce the aggregate travel cost based on the previous day's route travel costs. That is, the aggregated travel cost in the system based on the previous day's route travel costs will decrease when the route flows change from day to day. And also the day-to-day stationary link flows coincide with the DUE flow pattern. Yang and Zhang (2009) showed that the above-mentioned five categories of dynamical systems based on route flow variables satisfy the RBAP.

In this paper, we propose a discrete dynamical system model within a general framework of adjustment process called a discrete rational adjustment process (DRAP), to formulate the evolution of link flows from day to day in traffic networks. The proposed discrete dynamical system model with the DRAP complements the continuous dynamical system model investigated recently by Guo et al. (2013), but its convergence properties are substantially different. The rest of the paper is organized below. In Section 2, we propose the discrete dynamical system model formulating the day-to-day adjustment process of link flows towards the DUE state in a network with fixed demand, and establish theoretically a number of its desirable properties. Section 3 demonstrates the dynamical system model and its properties by numerical experiments. Finally, Section 4 concludes the paper and two appendixes are included to show a straightforward extension to the elastic demand case and the stochastic user equilibrium (SUE) case.

## 2. Adjustment process in a network with fixed demand

### 2.1. Model description

Consider a general traffic network $G(N, L)$ with a set $N$ of nodes and a set $L$ of directed links. Let $W$ be the set of OD pairs and $R_w$ be the set of routes connecting OD pair $w \in W$.

In this section, it is assumed that the travel demands in the network are fixed and denoted by a column vector $\mathbf{d} = (d_w, w \in W)^{\mathrm{T}}$, where $d_w$ is the travel demand between OD pair $w \in W$. The vector of route flows is denoted as $\mathbf{f} = (f_{r,w}, r \in R_w, w \in W)^{\mathrm{T}}$ with $f_{r,w}$ being the flow on route $r \in R_w$, and the vector of link flows is denoted as $\mathbf{x} = (x_a, a \in L)^{\mathrm{T}}$, with $x_a$ being