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Behavior of the cell transmission model and effectiveness of ramp metering

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Abstract

The paper characterizes the behavior of the cell transmission model of a freeway, divided into N sections or cells, each with one on-ramp and one off-ramp. The state of the dynamical system is the N-dimensional vector n of vehicle densities in the N sections. A feasible stationary demand pattern induces a unique equilibrium flow in each section. However, there is an infinite set—in fact a continuum—of equilibrium states, including a unique uncongested equilibrium n^u in which free flow speed prevails in all sections, and a unique most congested equilibrium n^{con}. In every other equilibrium n^e one or more sections are congested, and $n^u \leqslant n^e \leqslant n^{con}$. Every equilibrium is stable and every trajectory converges to some equilibrium state.

Two implications for ramp metering are explored. First, if the demand exceeds capacity and the ramps are not metered, every trajectory converges to the most congested equilibrium. Moreover, there is a ramp metering strategy that increases discharge flows and reduces total travel time compared with the no-metering strategy. Second, even when the demand is feasible but the freeway is initially congested, there is a ramp metering strategy that moves the system to the uncongested equilibrium and reduces total travel time. The two conclusions show that congestion invariably indicates wastefulness of freeway resources that ramp metering can eliminate.

Keywords: Cell transmission model; Stability; Macrosimulation; Ramp metering; Fundamental diagram; Congestion; Strictly monotone map; Convergent system

1. Introduction

The paper presents a complete analysis of the qualitative properties of the cell transmission model (CTM). CTM is a first-order discrete Godunov approximation (Godunov, 1959), proposed by Daganzo (1994) and Lebacque (1996), to the kinematic wave partial differential equation of Lighthill and Whitham (1955) and Richards (1956). The popularity of CTM is due to its very low computation requirements compared with

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micro-simulation models; the ease with which it can be calibrated using routinely available point detector data (Lin and Ahanotu, 1995; Munoz et al., 2004); its extensibility to networks (Buisson et al., 1996a) and urban roads with signalized intersections (Lo, 2001; Almasri and Friedrich, 2005); and the flexibility with which it can be used to pose questions of traffic assignment (Buisson et al., 1996b; Ziliaskopoulos, 2000) and ramp metering (Daganzo and Lin, 1993; Zhang et al., 1996; Gomes and Horowitz, 2006). These topics are also studied using different discrete models (May, 1981; Papageorgiou et al., 1990; Payne, 1979). CTM is a widely used discrete macroscopic model today.

The objective of this paper, however, is not to relate CTM to the kinematic wave equation nor to investigate its utility for simulation, but rather to study it as a class of nonlinear dynamical systems. From this viewpoint, the interest is to determine the structure of its equilibrium points, their stability, and the qualitative properties of the convergence of its trajectories. Surprisingly, these aspects of the cell transmission model have received no attention in the published literature. The paper fills this gap.

Section 3 presents the model, taken from Gomes (2004) and Gomes and Horowitz (2006), which in turn is based on Daganzo (1994). The freeway is divided into N sections, indexed $0, \ldots, N-1$. Section i is characterized by a single state variable, its density n_i, so the state of the freeway is the N-dimensional vector $n = (n_0, \ldots, n_{N-1})$. Vehicle movement in a section is governed by the familiar triangular 'fundamental diagram', which gives flow as a function of vehicle density. If the density is below critical, vehicles move at free flow speed; if it is above critical, the section is congested, speed is lower, and flow from the immediately upstream section is constrained. Thus the state of a freeway obeys a N-dimensional nonlinear difference equation. When the exogenous demand pattern of on-ramp and off-ramp flows is constant, the difference equation is time-invariant, and it is meaningful to study its equilibrium states.

Theorem 4.1 in Section 4 fully characterizes the structure of the equilibrium flows and states in any CTM model. Each demand pattern induces a unique equilibrium flow vector f, and an *infinite* set of equilibrium states E. Corresponding to f is the set of bottleneck sections at which flow equals capacity. If there are K bottlenecks, the freeway partitions into $1 + K$ segments, $S^0, \ldots, S^K$, each of which, except S^0, begins at a bottleneck, and decomposes the set E into the product $E = E^0 \times \cdots \times E^K$, with E^k being the equilibrium set for segment S^k. Each equilibrium in E^k determines an integer j so that the most downstream j sections in S^k are congested (density is above critical), and the remaining sections are uncongested. The equilibrium set E forms a topologically closed, connected, K-dimensional surface in the N-dimensional state space. Two equilibria are special: the unique *uncongested* equilibrium n^{u} in which free flow speed prevails in all sections; and the unique *most congested* equilibrium n^{con}. Every other equilibrium $n^{\mathrm{e}} \in E$ is bounded by these, $n^{\mathrm{u}} \leqslant n^{\mathrm{e}} \leqslant n^{\mathrm{con}}$. Theorem 4.1 provides an explicit closed form expression for E. In the special case that the demand is *strictly* feasible, i.e., the equilibrium flow in each section is strictly below capacity, E reduces to the unique uncongested equilibrium n^{u}.

Section 5 studies the qualitative behavior of all trajectories generated by the CTM model. The model induces a strictly monotone map and some of the trajectory behavior is a consequence of the general theory of strictly monotone maps (Hirsch and Smith, 2005). One interesting consequence is that the unique equilibrium n^{u} for a strictly feasible demand pattern is globally asymptotically stable: every trajectory converges to n^{u} (Theorem 5.1). A more surprising result is that every equilibrium is (Lyapunov) *stable*: trajectories starting near an equilibrium $n^{\mathrm{e}} \in E$ remain near it forever (Theorem 5.2). The most remarkable result is that the CTM model is *convergent*: every trajectory converges to some equilibrium state $n^{\mathrm{e}} \in E$ (Theorem 5.3).

Section 6 explores two implications for ramp metering. First, if the demand is infeasible and there is no-metering, every trajectory converges to the most congested equilibrium. However, there is a ramp metering strategy that increases flow in every section and reduces total travel time (Theorem 6.1). Second, even with feasible demand if the freeway is in a congested equilibrium, there is a ramp metering strategy that moves the freeway to the uncongested equilibrium while reducing total travel time (Theorem 6.2).

Section 7 summarizes the most significant results. Some proofs are collected in the Appendix.

2. Background of CTM

The simplest continuous macroscopic model was proposed by Lighthill and Whitham (1955) and Richards (1956), hence called LWR. Based on the conservation of vehicles, LWR is described by a single partial