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Topological maps and robust hierarchical Euclidean skeletons in cubical complexes

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ABSTRACT

Skeletons are notoriously sensitive to contour noise, and an effective filtering scheme is needed in any practical situation, where skeletons are involved. In this article, we introduce a new discrete framework that allows us to define and compute families of filtered Euclidean skeletons, in 2D as well as in 3D or higher dimensions. We prove several properties of our skeletonization scheme, in particular the preservation of topological characteristics and the stability with respect to parameter changes.

1. Introduction

Skeleton is one of the most studied and used concepts in pattern recognition and analysis. Since its introduction by Blum in the sixties [10], it has been the subject of hundreds of publications dealing with both practical and theoretical aspects. Indeed, despite the simplicity of its most common definition, as the set of all centers of maximal included balls, its use in real applications often raises difficult problems.

These difficulties are mainly due to two distinct issues.

First, the nice properties of skeleton that can be proved in the continuous framework (uniqueness, thinness, homotopy equivalence, invariance w.r.t. isometries) [28,25] do not all hold in discrete grids which are commonly used in image processing. Considerable effort has been devoted to design discrete skeletonization methods that aim at retrieving these properties, at least partially. These methods find their roots in different frameworks: discrete geometry [11,23,27,32,24], digital topology [19,40,39,31], mathematical morphology [33,37], computational geometry [2,3,29], and partial differential equations [35]. Recent surveys of the state of the art in skeletonization may be found in [17,36,8,9].

Second, even in the continuous framework the skeleton suffers from its sensitivity to small contour perturbations, in other words, its lack of stability. A recent survey [1] summarizes selected relevant studies dealing with this topic. This difficulty can be expressed mathematically: the transformation which associates a shape to its skeleton is only semi-continuous. This fact, among others, explains why it is usually necessary to add a filtering step (or pruning step) to any method that aims at computing the skeleton. Hence, there is a rich literature devoted to skeleton pruning, in which different criteria were proposed in order to discard "spurious" skeleton points or branches: see [4,29,3,27,2,38,24,5,18,26], to cite only a few.

Fig. 1 illustrates the four most popular ones among these criteria. Consider a skeleton point and its corresponding maximal ball (or disc in 2D), the most obvious criterion is based on the radius of this ball (a): the skeleton point is filtered out if this radius is beyond a given threshold. For defining the second criterion (b) and the following ones, we have to consider the projections of the skeleton point on the object boundary, that is, the contact points between the corresponding maximal ball and the boundary. The angle formed by these projections and the skeleton point as vertex, called bisector angle by some authors, also constitutes an effective filtering criterion [39,18].

If we consider now the distance between the projected points, when there are only two of them, or more generally the diameter[1] of the smallest ball that contains all these points (see Fig. 1c), we obtain the parameter λ studied by Chazal and Lieutier [14], which has interesting properties in relation with stability. These authors intro-

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[1] Equivalently, one can consider the radius instead of the diameter.

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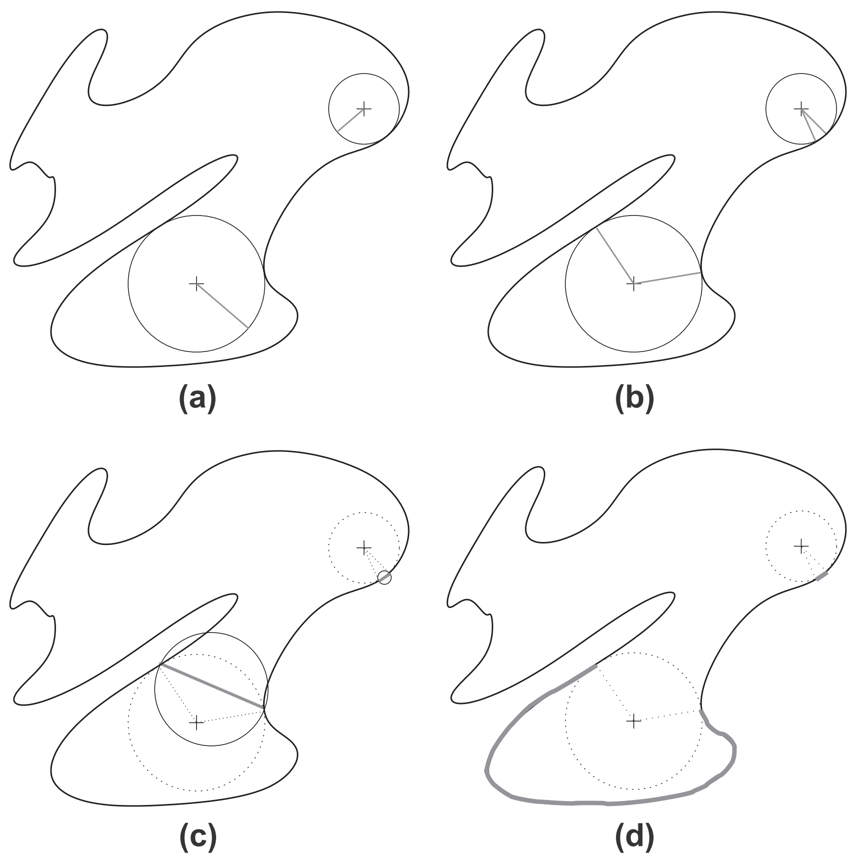

Fig. 1. Four criteria for filtering skeleton points: (a) radius, (b) bisector angle, (c) projection diameter, and (d) border portion length.

duced a particular class of filtered skeletons, called λ-medial axes, and they proved that small perturbations (in the sense of the Hausdorff distance) of the shape provoke only small perturbations of the skeleton, except for some critical values of λ. A discrete version of the λ-medial axis has been introduced and studied in [13], where its robustness to noise and its low sensitivity to rotations have been shown experimentally.

However, there are applications, where the presence of the critical values of λ is prohibitive. It is the case when the needed filtering level is equal to, or close to a critical value. In such situations, small changes of the filtering parameter may result in changes of the topological characteristics (*e.g.* the connectedness), or in sudden elimination or apparition of skeleton branches.

Let us illustrate this problem with the help of Fig. 2. In Fig. 2a, we see that the parameter value $\lambda = 2$ is not sufficient to filter out spurious branches of the λ-medial axis. However if we set $\lambda = 3$, we loose a big and meaningful skeleton branch, whereas some spurious branches are still present.

In 2D, this problem may be avoided by using a fourth criterion, which consists of measuring the length of the portion of the object boundary between the projected points, as illustrated in Fig. 1d. Based on this idea, several methods have been proposed: hierarchic skeletons [29], veinerization [30], multiscale skeletons [21]. The parameter for these methods is a threshold value for the border portion length criterion. It can be easily seen that small variations of this parameter do not provoke big changes in the obtained result, contrarily to what happens with the parameter λ.

Using any of these four criteria, one obtains for any object a family of nested skeletons, indexed by parameter values. Another way of seeing this family, is to consider the function that associates, to each object point, the value of the considered criterion. For example, the function on which is based the λ-medial axis is called *PR* (for Projection Radius) in this article. Final skeletons are obtained as level sets (*i.e.*, thresholds) of this function (see Fig. 6).

The aim of this article is to formalize and generalize, in a discrete framework, the approaches based on the fourth criterion (border portion length), for they provide the best stability with respect to variations of the filtering parameter. The method of R.L. Ogniewicz and O. Kübler [29] is defined in the framework of the 2D constinuous plane, more precisely it applies to (sets of) planar polygons, and the resulting skeletons are made of straight line segments. These skeletons are proved to be homotopy-equivalent with inital objects, however if one needs to discretize these skeletons in $\mathbb{Z}^2$, one looses this property. On the other hand, the methods proposed by Pierrot-Deseilligny et al. [30] and Falcao et al. [21] are defined in the 2D square grid. However [30] does not provide an algorithm to compute skeletons in practice, and the algorithm proposed in [21] does not guarantee topology preservation.

The discrete objects that we consider in this article are cubical complexes, that is, they are sets of elements of different dimensions (points, segments, squares, cubes, etc.) that are glued together according to certain rules (see Section 2). We consider here 2D and 3D cubical spaces, however our approach extends easily to any finite dimension.

The first step of our skeletonization scheme consists of a directional parallel thinning (Section 5), guided by the priority function *PR* (Section 4), and based on the operation of collapse (Section 3). Collapse is an elementary topology-preserving transforma-