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The minimum barrier distance

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ABSTRACT

In this paper we introduce a minimum barrier distance, MBD, defined for the (graphs of) real-valued bounded functions f_A, whose domain D is a compact subsets of the Euclidean space $\mathbb{R}^n$. The formulation of MBD is presented in the continuous setting, where D is a simply connected region in $\mathbb{R}^n$, as well as in the case where D is a digital scene. The MBD is defined as the minimal value of the barrier strength of a path between the points, which constitutes the length of the smallest interval containing all values of f_A along the path.

We present several important properties of MBD, including the theorems: on the equivalence between the MBD ρ_A and its alternative definition φ_A; and on the convergence of their digital versions, $\widehat{\rho_A}$ and $\widehat{\varphi_A}$, to the continuous MBD $\rho_A = \varphi_A$ as we increase a precision of sampling. This last result provides an estimation of the discrepancy between the value of $\widehat{\rho_A}$ and of its approximation $\widehat{\varphi_A}$. An efficient computational solution for the approximation $\widehat{\varphi_A}$ of $\widehat{\rho_A}$ is presented. We experimentally investigate the robustness of MBD to noise and blur, as well as its stability with respect to the change of a position of points within the same object (or its background). These experiments are used to compare MBD with other distance functions: fuzzy distance, geodesic distance, and max-arc distance. A favorable outcome for MBD of this comparison suggests that the proposed minimum barrier distance is potentially useful in different imaging tasks, such as image segmentation.

1. Introduction

Over the past several decades, distance transform (DT) [1–7] has been widely used as an effective tool for analyzing object morphology and geometry [8–10]. Most DT measures described in the literature essentially capture the Euclidean distance of a candidate point from a target set, often the background. Rosenfeld and Pfaltz [11] introduced the simple yet fundamental idea that, in a digital grid, the global Euclidean distance transform may be approximated by propagating local distances between neighboring pixels. Borgefors [2,3] extensively studied DTs for binary objects including the popular algorithm [2] that computes DT by using different local step lengths for different types of neighbors. Also, she studied the geometry and equations of 3D DT and presented a two-pass raster scan algorithm for computing approximate Euclidean distance transform [3]. An algorithm for computing in linear time the exact Euclidean distance transform for the rectangular digital images was described in [12] and elaborated on in [13].

Other authors have considered distance functions where the image data is taken into account, see, e.g., [5,14–16]. Distance transforms for such distance functions are typically computed on discrete sets using variations on Dijkstra's algorithm. Falcão et al. showed that this method of computation can be used for any *smooth* distance function, as defined in [16].

Image processing on fuzzy subsets has gained a lot attention, [9,10,17,18]. It provides a flexible framework for handling uncertainty, arising from sampling artifacts, illumination inhomogeneities and other imperfections in the image representation and acquisition process. Fuzzy sets are defined using a membership function which gives the degree of belongingness with respect to some set.

In this paper, we introduce a distance function defined for the real-valued bounded functions f_A (so, in particular, for fuzzy sets), whose domain D is a compact subsets of the Euclidean space $\mathbb{R}^n$. We refer to the new distance as the "minimum barrier distance" and study its properties in the continuous setting, where D is a simply connected region in $\mathbb{R}^n$, as well as in the case where D is a digital scene. In image processing and computer vision, ordinary and fuzzy distance functions [1–6,11,14–16] have widely been used to represent a spatial relation between each pair of points in a Euclidean space or a fuzzy subset. For example, ordinary dis-

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tance function, commonly used for binary images, is a measure of the shortest digital path length between two points while, as viewed by Saha et al. [5], fuzzy distance is a measure of the "minimum material to be traversed" to move from one point to the other where the fuzzy membership function is linked to local material density. Under both ordinary and fuzzy distance frameworks, the length of a path strictly increases as the path grows. The formulation of minimum barrier distance function possesses the following property: The length of a path may remain constant during its growth until a new stronger barrier is met on the path. This subtle shift in the notion of path length allows the new distance function to capture separation between two points in the sense of "connectivity" [19] in a fuzzy set unlike geometric properties commonly represented by existing distance functions. Thus, it may be an interesting avenue to study strengths and limitations of the new function that theoretically behaves like a distance while resembling to "anti-connectivity" from a user perspective. For example, the new distance may be useful to determine minimum barrier to move from one region to another and also, to locate the minimum barrier path. In the context of image processing and computer vision, the new distance function may be useful in image segmentation and region growing.

We show that (pseudo-)metric properties of the "minimum barrier distance" are maintained by its formulation in a digital grid. We give examples that show that the minimum-barrier distance cannot be computed using the standard Dijkstra algorithm mentioned above. Instead, we give an approximation of the new distance measure for fuzzy subsets on digital grids and show that the minimum barrier distance over a continuous fuzzy subset can be approximated arbitrarily close in a digital grid by using a sufficiently dense sampling grid. A similar approximation idea is presented in [20]. An efficient computational solution for the minimum barrier distance is presented using the approximation. The experiments show that the minimum barrier distance is robust to noise, blur, and seed point position.

2. The Minimum Barrier Distance in R^n

Let $f_A : D \to \mathbb{R}$ be any bounded function and let A be its graph, that is,

$$A = \{\langle x, f_A(x)\rangle : x \in D\}.$$

We will concentrate on the functions f_A: $D \to [0,1]$, in which case A will be treated as a fuzzy subset of D and f_A will be referred to as the *membership function* of A in D. However, the presented material works for mappings f_A with any bounded range. For example, $f_A(x)$ could be the intensity value at x in a digital image.

For $D \subset \mathbb{R}^n$ and $p,q \in D$, a *path from p to q* (in D) is any continuous function π: $[0,1] \to D$ with $p = \pi(0)$ and $q = \pi(1)$. We use the symbol $\Pi_{p,q}$ (or just Π, when p and q are clear from the context) to denote the family of all such paths. Recall, that $D \subset \mathbb{R}^n$ is *path connected* provided for every $p,q \in D$ there exists a path π: $[0,1] \to D$ from p to q.

The goal of this section is to introduce and discuss the following notion of the minimum barrier distance defined for the bounded <u>continuous</u> functions $f_A : D \to \mathbb{R}$ in the case when $D \subset \mathbb{R}^n$ is path connected.

Definition 1. For a path π: $[0,1] \to D$, the *barrier* along π is defined as

$$\tau_A(\pi) = \max_t f_A(\pi(t)) - \min_t f_A(\pi(t)) = \max_{t_0,t_1}(f_A(\pi(t_1)) - f_A(\pi(t_0))). \quad (1)$$

The *minimum barrier distance* ρ_A: $D \times D \to [0,\infty)$ for a path connected $D \subset \mathbb{R}^n$ is defined via formula

$$\rho_A(p,q) = \inf_{\pi \in \Pi_{p,q}} \tau_A(\pi). \quad (2)$$

Notice that the maxima and minima in the formula (1) are attained (by the Extreme Value Theorem), since the composition function $f_A \circ \pi$ is continuous. At the same time, the next example shows that a path that defines the minimum barrier distance ρ_A is not always attained, that is, the infimum operation in the definition (2) cannot be replaced with the minimum operation.

Example 1. Let $D = [-1,1]^2$ and T be the topologists sine curve, that is, T is the closure of the set $S = \{\langle x, \sin(1/x)\rangle : x \in (0,1]\}$, see Fig. 1. If $f_A(x)$ is defined as the Euclidean distance from $x \in D$ to T, then f_A is continuous. If $p = \langle 0,0\rangle$ and $q = \langle 1, \sin 1\rangle$, then $\inf_{\pi \in \Pi_{p,q}} \tau_A(\pi) = 0$, but $\tau_A(\pi) > 0$ for any $\pi \in \Pi$ (since T is not path connected).

Notice that $\rho_A(p,q)$ is related to the geodesic distance $g_A(p,q)$ between the points $\langle p, f_A(p)\rangle$ and $\langle q, f_A(q)\rangle$ along the surface A. Actually, $\rho_A(p,q)$ is, in a way, a vertical component of $g_A(p,q)$, so that $\rho_A(p,q) \leqslant g_A(p,q)$.

Definition 2. A function d: $D \times D \to [0,\infty)$ is a *metric* on a set D provided, for every $x,y,z \in D$,

(i) $d(x,x) = 0$ (identity)
(ii) $d(x,y) > 0$ for all $x \neq y$ (positivity)
(iii) $d(x,y) = d(y,x)$ (symmetry)
(iv) $d(x,z) \leqslant d(x,y) + d(y,z)$ (triangle inequality)

A function d that obeys properties (i), (iii), and (iv) is called a *pseudo-metric*.

In the proof that ρ_A is a pseudo-metric, we will use the following notion. The *concatenation* $\pi_1 \cdot \pi_2$ of the paths π_1 and π_2 such that $\pi_1(1) = \pi_2(0)$ is

$$(\pi_1 \cdot \pi_2)(t) = \begin{cases} \pi_1(2t) & \text{if } t \in [0,1/2] \\ \pi_2(2t) & \text{otherwise.} \end{cases}$$

Remark 1. If $\pi_1(1) = \pi_2(0)$, then $\tau_A(\pi_1) + \tau_A(\pi_2) \geqslant \tau_A(\pi_1 \cdot \pi_2)$.

Proposition 1. *ρ_A is a pseudo-metric.*

Proof. It is obvious that ρ_A is non-negative and symmetric. It satisfies the identity property (i), since for the constant path π_x defined via $\pi_x(t) = x$ for all $t \in [0,1]$, we have $\rho_A(x,x) \leqslant \tau_A(\pi_x) = f_A(x) - f_A(x) = 0$.

Now we prove the triangular inequality. Given three arbitrary points $p,q,r \in D$ and an $\varepsilon > 0$ chose the paths $\pi_{p,q} \in \Pi_{p,q}$ and $\pi_{q,r} \in \Pi_{q,r}$ such that $\rho_A(p,q) \geqslant \tau(\pi_{p,q}) - \varepsilon$ and $\rho_A(q,r) \geqslant \tau(\pi_{q,r}) - \varepsilon$. Then, using Remark 1, we have

$$\rho_A(p,q) + \rho_A(q,r) \geqslant \tau(\pi_{p,q}) - \varepsilon + \tau(\pi_{q,r}) - \varepsilon \geqslant \tau_A(\pi_{p,q} \cdot \pi_{q,r}) - 2\varepsilon \geqslant \rho_A(p,r) - 2\varepsilon.$$

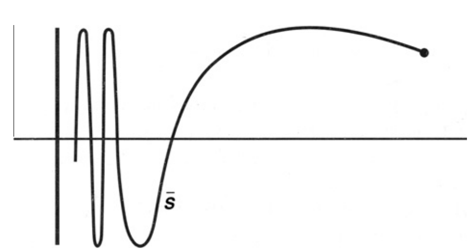

Fig. 1. Topologists sine curve from Example 1.