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Topological analysis of shapes using Morse theory ☆

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Abstract

In this paper, we propose a novel method for shape analysis that is suitable for any multi-dimensional data set that can be modelled as a manifold. The descriptor is obtained for any pair (M, φ), where M is a closed smooth manifold and φ is a Morse function defined on M. More precisely, we characterize the topology of all pairs of sub-level sets (M_y, M_x) of φ, where $M_a = \varphi^{-1}((-\infty,\ a])$, for all $a \in \mathbb{R}$. Classical Morse theory is used to establish a link between the topology of a pair of sub-level sets of φ and its critical points lying between the two levels.

Keywords: Shape representation; Shape similarity; Morse theory; Computational homology

1. Introduction

In computer vision, the problems of shape recognition, classification, and matching require adequate tools to represent shape properties by means of descriptors. Such descriptors allow to measure several shape characteristics of statistical, geometrical, or topological nature. In this paper we focus on topological descriptors since they provide information that can remain constant despite the variability in appearance of objects due to noise, deformation and other distortions. They also allow significant reduction in the amount of data while providing sufficient information to characterize and represent objects. We construct topological descriptors for objects modelled as smooth manifolds which are based on the knowledge of Morse functions that measure some metric properties of the given objects. So our descriptors are really built of a marriage between the geometric and topological properties of the objects under consideration.

More precisely, our descriptors are obtained as follows. Consider a pair (M, φ), where M is a closed smooth manifold and φ is a Morse function defined on M. We denote by M_t the sub-manifold of M also called the sub-level set of φ and consisting of all points of M at which φ takes values less than or equal to t, i.e., $M_t = \{p \in M | \varphi(p) \leqslant t\}$. We associate to the pair (M, φ) a structure that encodes the topology of all pairs of sub-level sets (M_y, M_x) of φ, as x, $y (x \leqslant y)$ vary in $\mathbb{R}$. We want to emphasize the fact that we apply homology not to the manifold M itself, but to derived spaces that have richer geometric content. In this case, we construct spaces out of M by using sub-levels of a Morse function on M.

The framework of classical Morse theory [9,8] allows to establish a link between the topology of a given pair of sub-level sets of φ and its critical points lying between the two levels. A number of results in this theory prove that the changes in topology of M are intimately related to the presence of critical points of some Morse function on M. For instance, the well-known Morse inequalities provide a lower bound for the number of critical points of index λ of a Morse function on M in terms of the Betti number $\beta_\lambda(M)$, which is determined by the topological shape of M. Moreover, the assumptions on M being closed smooth manifold and φ is a Morse function allow to justify

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considering only a finite number of sub-level sets and, hence, obtaining a discrete but satisfactory description of the shape of M. In practice, one can include open connected surfaces and bounded Morse functions.

We use relative homology groups to compute the topology of pairs of sub-level sets of a given pair (M, φ). The different levels of relative homology groups of the whole structure of sub-level sets encode complete topological information about the critical regions of the manifold as well as the extension of the regions where no topological change is observed. For the experimental results, our manifold is given in a form of a simplicial complex. In other words, we assume that our manifold M is the support of a simplicial complex K (see Section 2.2 for an overview on cell-complexes and homology) and a special care is taken in the sampling process to ensure that all sub-level sets of the manifold are sub-complexes of K. The final structure consists of matrices each of which is encoding a different level of relative homology. Two structures associated to two pairs (M_1, φ_1) and (M_2, φ_2) are compared by defining an appropriate distance function between the two collections of associated matrices.

Our structure allows to recover two very well-known topological descriptors, namely the size functions [5,13] and the presentations of homology groups in [4]. Size functions are defined in terms of the number of connected components of sub-level sets associated with a given curve and a suitable measuring function defined on the curve. They can easily be reformulated with the concepts of homology and relative homology as done in [2].

First, we give a brief overview of some techniques from algebraic topology. Then we follow with a section on the basic concepts of Morse theory. In Section 3, we introduce the main definitions and characteristics of the new shape descriptor. Computational results are presented in Section 4 followed by a brief conclusion in the last section.

2. Brief overview of algebraic topology

In this section, we give a brief review of basic notions and terminology of algebraic topology. The discussion will be informal and is intended to give an idea of what these notions are about. For a more complete description, we refer the reader to any standard text in the area, such as [6,10,12], or a more recent text [7] that uses a computer to develop a combinatorial computational approach to the subject.

2.1. Homotopy

A very important notion in algebraic topology is that of a homotopy of maps between topological spaces. Two maps f, g: $X \to Y$ are said to be homotopic if there is a continuous map h: $X \times [0,1] \to Y$, called a homotopy between f and g, so that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all $x \in X$. We write $f \simeq g$ whenever f is homotopic to g. The map h is called a *homotopy* of f to g. We may think of h as a way of "deforming" f continuously to g, as t varies from 0 to 1. For example, any two continuous maps $f, g : X \to \mathbb{R}^n$ are homotopic for any topological space X. The formula $h(t,x) = (1-t)f(x) + tg(x)$ is called the straight-line homotopy between them. A map f: $X \to Y$ is a *homotopy equivalence* if there is a map g: $Y \to X$ so that $f \circ g$ is homotopic to the identity map on Y and $g \circ f$ is homotopic to the identity map on X. This is a weakening of the notion of *homeomorphism* that requires $f \circ g$ and $g \circ f$ to be equal to the corresponding identity maps. Two spaces X and Y are said to be *homotopy equivalent*, or to have the same *homotopy type*, if there is a homotopy equivalence from X to Y. This is denoted by $X \simeq Y$. Let $A \subset X$. A *retraction* of X onto A is a continuous map r: $X \to A$ such that $r(a) = a$ for each $a \in A$. We then say that A is a *retract* of X. A *deformation retraction* of X onto A is a continuous map h: $X \times [0,1] \to X$ such that $h(x,0) = x$ for $x \in X$, $h(x,1) \in A$, for all $x \in X$, and $h(a,t) = a$, for all $a \in A$, and $t \in [0,1]$. If such an h exists, we say that A is a *deformation retract* of X. If A is a deformation retract of X, then A and X are homotopy equivalent. Indeed, if i denotes the inclusion of A into X and $r(x) = h(x,1)$, then $r \circ i$ is equal to the identity map on A, and $i \circ r$ is homotopic to the identity on X.

Example 2.1. Let X be the annulus shown in Fig. 1. We parameterize it with polar coordinates $(r, \theta), 1 \leqslant r \leqslant 2$, $0 \leqslant \theta < 2\pi$. Let A be the unit circle and f be the map from X to A given by $f(r,\theta) = (1,\theta)$. Then f is a homotopy equivalence, since the inclusion of the circle into X provides the required map g. It is easily seen that f is a retraction from X onto A and the map $h(r,\theta,t) = t f(r,\theta) + (1-t)(r,\theta)$ defines a deformation retraction of the annulus onto the unit circle.

2.2. Homology

In most cases, it is rather complicated to decide when two maps between spaces are homotopic, or two spaces are homotopy equivalent. In applications where a topological tool is needed to compare between spaces and maps, one is usually satisfied in comparing their homology structures which are coarser but computable. Although homology is less intuitive than homotopy, its combinatorial

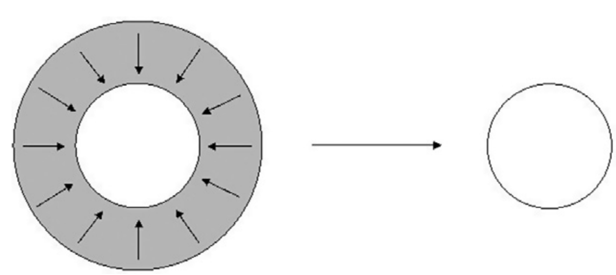

Fig. 1. A circle is a deformation retract of the annulus.